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ECONOMIC GENERALITY VERSUS MATHEMATICAL GENERICITY: ACTIVITY-LEVEL INDETERMINACY AND THE INDEX THEOREM IN CONSTANT RETURNS PRODUCTION ECONOMIES

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ABSTRACT

When the mathematical concept of genericity was arrived at in economics, it was meant more or less as a synonym for generality. Referring to constant return production economies, we will argue that this is not always the case. In particular, the representation of technology that is mathematically generic is not at all general for economists. We will see that in cases that are economically general, but not mathematically generic, activity-level indeterminacy may occur. In these cases, Kehoe's index theorem, a well-known result of the application of the differentiable approach to production economies, becomes unusable.

1. INTRODUCTION

Let $g(p)$ be a smooth function whose fixed points $\hat{p} = g(\hat{p})$ are equivalent to equilibria of an economy; according to a well-known terminology (Debreu, 1970, 1976), the economy is called 'regular' if the matrix $[I - Dg(\hat{p})]$ has non-zero determinant at every fixed point.

Thanks to differential topology, we know that regular economies have a finite and odd number of equilibria. In fact, let F be the set having as elements fixed points of the smooth function $g(p)$, the fixed point index theorem states that $\Sigma_{\hat{p}\in F}\text{index}(\hat{p}) = +1$, where: index $(\hat{p})$ = sign det$[I - Dg(\hat{p})]$. Therefore, a regular economy has $2k + 1$ equilibria, where k is the number of equilibria with index -1 (Dierker, 1972; Varian, 1975).

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In order to use this approach, the map $g(p)$ must be not only continuous—a property sufficient for the existence of at least a fixed point—but also differentiable at its fixed points, otherwise matrix $Dg(\hat{p})$ does not exist and the method breaks down.

The main aim of the present paper is to discuss the specific assumptions that have been used for the extension of this method to production economies. In particular, we will argue that the assumptions assuring differentiability of $g(p)$ at its fixed points may force us to consider only very peculiar models of production.

We introduce the argument by studying a very simple example of economy with constant returns production, and show that the presence of free disposal activity—necessary for the existence of at least an equilibrium—may bring about activity-level indeterminacy. Although activity-level indeterminacy is not a problem in itself, we shall see that it prevents the use of some differentiable approach's tools. In fact, if equilibrium activity levels are indeterminate, then the function whose fixed points are equivalent to equilibria of the economy is not differentiable at its fixed points.

As a consequence, in constant returns production economies, the use of the tools mentioned above requires specific assumption on technology. These specific assumptions are generically satisfied when technical coefficients in the matrix of activities are chosen randomly, but, as we will show in this paper, they are not satisfied in the cases that economists consider as general. In particular, we will argue that the matrices of activities that are generic, in a mathematical sense, correspond to a representation of technology such that each production activity, if in use, gives strictly positive quantities of all the outputs of the economy and uses strictly positive quantities of all the inputs.

2. THE MODEL

We assume there to be n different commodities, where differences between commodities depend not only on their physical characteristics but also on their date of delivery.

The consumption side is described by an aggregate excess demand function $z : R^n_{++} \to R^n$ satisfying the following assumptions.

Assumption 1 (homogeneity): Each $z_i(p)$ in the vector function $z(p) = [z_1(p), \ldots, z_n(p)]$ is homogeneous of degree zero, i.e. $z_i(tp) = z_i(p)$ for every scalar $t > 0$.

Assumption 2 (Walras' law): The vector function $z(p)$ satisfies Walras' law, i.e. $p'z(p) \equiv 0$.

Assumption 3 (differentiability): Each $z_i(p)$ is continuously differentiable on its domain.

Assumptions 1 and 2 are conventional and do not require any comment. On the other hand, differentiability, rather than simple continuity, is a specific assumption needed in order to use differential topology tools.

As regards production, technology is specified by an activity matrix A, with a number of rows equal to the number of commodities n, and a number of columns m equal to the number of different production activities. Positive entries in A represent output coefficients; negative entries are for input coefficients. Let $v \in R_+^n$ be a vector of activity levels, the vector $y = Av$ is a technically feasible aggregate production plan.

The assumptions we posit on activity matrix A are as follows.

Assumption 4 (free disposability): A includes n free disposal activities, one for each commodity. In other words: $A = [H \,\vdots\, -I_n]$, where H is a matrix of $n \times (m - n)$ dimension, having as columns the coefficients of productive activities *strictu sensu*, and $-I_n$ is the matrix made up by free disposal activities.

Assumption 5 (boundedness): There are no outputs without any inputs, i.e. matrix A is such that $\{Av \in R^n : v \geq 0, Av \geq 0\} \equiv \{0\}$.

Assumptions 4 and 5 are conventional and play a role in assuring the existence of at least an equilibrium; while, as we will see later, stronger assumptions are needed in order to use differentiable methods.

Given an aggregate excess demand function $z(p)$ satisfying Assumption 1 and Assumption 2 and an activity matrix A satisfying Assumption 4 and Assumption 5, the pair (z, A) identifies an economy in the space E.

Let $S = \{p \in R_{++}^n : \Sigma p_i = 1\}$ be the normalized price simplex and let $S_A = \{p \in S : p'A \leq 0\}$ be the subset of S containing price vectors that satisfy the non-positive extra-profits condition, then an equilibrium for an economy (z, A) can be defined in the following terms.

Definition 1 (equilibrium): a price vector $\hat{p} \in S_A$ and a vector of activity levels $\hat{v} \in R_+^m$ are an equilibrium for the economy (z, A) if and only if $z(\hat{p}) = A\hat{v}$.[1]

[1] Note that $\hat{p} \in S_A$, Walras' law and the equilibrium condition $z(\hat{p}) = A\hat{v}$ imply that: $\hat{p}'Av \leq 0$ for every $v \in R_+^m$ and $\hat{p}'A\hat{v} = 0$. Therefore, the vector of activity levels $\hat{v}$ is profit maximizing at the price vector $\hat{p}$.

3. AN EXAMPLE OF ACTIVITY-LEVEL INDETERMINACY

Let us consider an economy with two products, which we call 1 and 2, and with two factors, which we call 3 and 4.

Let us suppose the matrix of activities of the economy is

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -a_{31} & -a_{32} & 0 & 0 & -1 & 0 \\ 0 & -a_{42} & 0 & 0 & 0 & -1 \end{bmatrix}$$

It can immediately be verified that matrix A set out above satisfies Assumption 4, and we assume that coefficients a_{ij} are strictly positive so that Assumption 5 is also satisfied.

Let us consider the price vector

$$\hat{p} = \left[0; \frac{a_{42}}{1 + a_{42}}; 0; \frac{1}{1 + a_{42}} \right]$$

and suppose that, at least in a neighbourhood of $\hat{p}$, the functions of market excess demand for the five commodities are

$$z_1(p) = \alpha; \quad z_2(p) = \frac{\omega_3 p_3 + \omega_4 p_4 - \alpha p_1}{p_2}; \quad z_3(p) = -\omega_3; \quad z_4(p) = -\omega_4$$

The market excess demand functions above are homogeneous of degree zero, satisfy 'Walras' law' and are differentiable.

Let θ be a parameter in the open interval (0; 1), it is possible to show that for every θ in (0; 1) there exists a vector of activity levels $\hat{v}(\theta)$ such that the price vector $\hat{p}$ and the vector of activity levels $\hat{v}(\theta)$ are an equilibrium for the economy we are considering. In other words, there is a continuum of vectors of activity levels for which the economy is in equilibrium.

In order to prove it, we can define, under weak conditions,[2] a vector $\hat{v}(\theta) = R_+^6$, for every $\theta \in (0; 1)$, by the following rule:

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[2] We assume that

$$\alpha \geq \frac{\omega_4}{a_{42}}$$

and

$$\omega_3 \geq \left(\alpha - \frac{\omega_4}{a_{42}} + 1 \right) a_{31} + \frac{\omega_4}{a_{42}} a_{32}$$

$$\hat{v}_1(\theta) = \alpha - \frac{\omega_4}{a_{42}} + \theta; \quad \hat{v}_2(\theta) = \frac{\omega_4}{a_{42}}; \quad \hat{v}_3(\theta) = \theta; \quad \hat{v}_4(\theta) = 0; \quad \hat{v}_5(\theta) = \omega_3 - \left(\alpha - \frac{\omega_4}{a_{42}} + \theta\right) a_{31} - \frac{\omega_4}{a_{42}} a_{32}; \quad \hat{v}_6(\theta) = 0$$

It is now immediate verifiable that $z(\hat{p}) = A\hat{v}(\theta)$ for every $\theta \in (0; 1)$.

This kind of activity-level indeterminacy arises because the activities earning zero profit at the equilibrium price vector $\hat{p}$ are not linearly independent. In particular, let $B(\hat{p})$ be the submatrix of A formed only by the columns referring to activities yielding zero profits at $\hat{p}$, thus:

$$B(\hat{p}) = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -a_{31} & -a_{32} & 0 & -1 \\ 0 & -a_{42} & 0 & 0 \end{bmatrix}$$

the matrix $B(\hat{p})$ columns are not linearly independent,[3] so that $\det[B(\hat{p})] = 0$.

Let v^b be a vector of activity levels for the four activities in $B(\hat{p})$, with $\hat{v}^b >> 0$, the linear system $z(\hat{p}) = B(\hat{p})v^b$ cannot uniquely determine the equilibrium vector of activity levels $\hat{v}^b$ as it has less than four independent equations. In section 5 we will see that the linear dependency of the $B(\hat{p})$ columns prevents the use of differentiable topology tools.

4. THE DIFFERENTIABLE APPROACH

Let $p^{S_A} : R^n \rightarrow S_A$ be the function that projects any vector of R^n into the space $S_A \subset S$, i.e. the function that associates any vector in R^n with its closest vector in S_A, we can define the following function.

Definition 2 (the mapping): For any economy (z, A), define the map $g : S \rightarrow S$ by the rule $g(p) = p^{S_A}(p + z(p))$.

Given the map $g(p)$, the following theorem has been proved (see Kehoe, 1980; see also Kehoe, 1982, 1985, 1991, 1998).

Theorem 1: Fixed points $\hat{p} = g(\hat{p})$ and equilibria of the economy (z, A) are equivalent.

[3] The first column of $B(\hat{p})$ can be expressed as a linear combination of the last two columns.

Thanks to Assumptions 1–5, $g(\hat{p})$ is a continuous map from the compact and convex set S to itself, therefore the Brouwer's fixed point theorem implies that at least an equilibrium exists.

In order to go further, i.e. in order to use differential topology tools for the study of local and global uniqueness, two more assumptions are required.

Assumption 6 (rank condition): No column of A can be expressed as a linear combination of fewer than n other columns. That is, however we take n columns of A, they must be linearly independent.[4]

Assumption 7: Let $B(p)$ be the submatrix of A whose columns are the activities earning zero profit a p, i.e. $p'B(p) = 0$. At every equilibrium price vector $\hat{p}$ all the activities in matrix $B(\hat{p})$ are in use, i.e. their corresponding levels are strictly positive.

With the help of these further assumptions, the following theorem has been proved (see Kehoe, 1980).

Theorem 2: For every economy (z, A) satisfying Assumptions 1–7, the map $g(p)$ is differentiable at least in a neighbourhood of every fixed point $\hat{p}$. Let $C = [e \,\vdots\, B(\hat{p})]$, where e is an $(n \times 1)$ vector of 1s and $B(\hat{p})$ the submatrix of A whose columns are all the activities earning zero profit at $\hat{p}$, then the Jacobian matrix of $g(p)$ at $\hat{p}$ is $Dg(\hat{p}) = (I - C(C'C)^{-1}C')(I + Dz(\hat{p}))$.

Therefore, production economies satisfying Assumptions 1–7 are generically regular, so that the fixed point index theorem implies that they have a finite and odd number of equilibria.

Moreover, let F be the subset of S having as elements fixed points of the function $g(p)$, if index$(\hat{p}) = +1$ for every $\hat{p} \in F$, then F is a singleton set.

5. ACTIVITY-LEVEL INDETERMINACY AND THE DIFFERENTIABLE APPROACH

In section 3, we saw an example of activity-level indeterminacy associated with linear dependency of $B(\hat{p})$ columns. When $B(\hat{p})$ has linearly dependent columns, matrix $C = [e \,\vdots\, B(\hat{p})]$ also has linearly dependent columns and the following lemma applies.

[4] In order to have some reference, we find this assumption stated explicitly in Kehoe (1980, p. 1218) and in Mas-Colell (1985, p. 249).

Lemma 1: Let C be an $(n \times c)$ *matrix, if columns of C are linearly dependent then* $\det(C'C) = 0$.[5]

As a consequence, in the case of the activity-level indeterminacy we are considering, the matrix $(C'C)$ is not invertible and the Jacobian matrix $Dg(\hat{p})$ does not exist. In this case, as the map $g(p)$ is not differentiable in a neighbourhood of fixed point $\hat{p}$, then equilibria at $\hat{p}$ would have no index, and no zero index either.[6]

Therefore, in order to assure the differentiability of $g(p)$ at its fixed points, it is necessary to posit assumptions ruling out activity-level indeterminacy. This is the role played by Assumption 6 (rank condition). In fact, as $\hat{p}'B(\hat{p}) = 0$ and $\hat{p} \neq 0$, Assumption 6 implies, together with the others, that $B(\hat{p})$ has fewer than n columns and these are linearly independent. In this case, the matrix $(C'C)$ is invertible and theorem 2 holds.

The problem is that Assumption 6, together with Assumption 4 (free disposability), implies very peculiar cases of production.

In order to grasp the problem, let us begin by remembering that, because of Assumption 4, the matrix A is the composition of two submatrices: matrix H, containing strictly productive activities; and matrix $-I_n$, containing n free disposal activities, one for each commodity.

Now, if matrix H has at least a zero entry—i.e. if there is at least a commodity that in some activity is neither an input nor an output—then matrix A does not satisfy Assumption 6, i.e. we can take n columns of A that are linearly dependent.

This can be proved easily: let us suppose, for instance, that the first element in the jth column of matrix H is $h_{1j} = 0$. Taking the jth column of H and the last $n - 1$ columns of matrix $-I_n$ we obtain the following square matrix:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ h_{2j} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{nj} & 0 & \cdots & -1 \end{bmatrix}$$

As the first row of this matrix is the zero vector, its determinant is zero and consequently its n columns are linearly dependent, then matrix A does not satisfy Assumption 6.

[5] Let c_{*i} be the ith column of C and let d_{*i} be the ith column of $D = (C'C)$; we have $d_{*i} = C'c_{*i}$. Now, if there are $c - 1$ or less nonzero coefficients α such that $c_{*\ell} = \Sigma_{i \neq \ell} \alpha_i c_{*i}$, then $d_{*\ell} = C'c_{*\ell} = C'\Sigma_{i \neq \ell} \alpha_i c_{*i} = \Sigma_{i \neq \ell} \alpha_i C' c_{*i} = \Sigma_{i \neq \ell} \alpha_i d_{*i}$. Therefore, when columns of C are linearly dependent, columns of $D = (C'C)$ are also linearly dependent. In this case, $\det(C'C) = \det(D) = 0$.

[6] The zero index is only for a continuum of equilibrium price vectors.

Therefore, the following is true.

Proposition 1: Let $A = [H \,\vdots\, -I_n]$, if one or more elements of matrix H are zero, then it is impossible for matrix A to satisfy Assumption 6.

We conclude that Assumption 6 can be satisfied only if matrix H has no zero entry.

6. ECONOMIC GENERALITY VERSUS MATHEMATICAL GENERICITY

As our example of section 3 shows, in production economies whose technology is represented by an activity matrix $A = [H \,\vdots\, -I_n]$, activity-level indeterminacy may very well occur. This is not a problem in itself, but it may prevent the differentiability of the function $g(p)$ at its fixed points. Therefore, for the extension of the index theorem to constant returns production economies, Kehoe uses a specific assumption—i.e. Assumption 6 (rank condition)—in order to avoid activity-level indeterminacy.[7]

In section 5 we have argued that Assumption 6 implies the absence of zero entries in matrix H. Of course, if we choose the technical coefficient in matrix H randomly, then generically the matrix H will not have zero entries;[8] therefore, from a mathematical point of view, Assumption 6 is generically satisfied.

However, what seems generic in mathematical terms may be extremely specific in economic terms. In fact, focusing the attention on cases in which matrix H has no zero entries forces us to consider only very special models of production, in which every strictly productive activity uses all the inputs of the economy, and gives jointly all the outputs of the economy.

This very special kind of joint production not only moves the theory away from reality, but may also be an economic nonsense because commodities delivered now can be hardly conceived as jointly produced with commodities delivered in 10 years. In the same way, inputs available in 10 years cannot play any role in the process of production of commodities delivered now. Moreover, even in a temporal production models, the presence of inputs that

[7] As we said above (cf. footnote 4), the same assumption is used by Mas-Colell too.

[8] The probability of extracting a given number, e.g. zero, from the set of all real numbers is of course zero.

are specific to some activity, and not used in the others, is very far from being inconceivable.

It is also worth mentioning that the case very common in literature, of several produced commodities without joint production, would be considered non-generic and ruled out by Assumption 6. There is, indeed, a relevant discrepancy between what is mathematically generic and what is interesting for economists.

In conclusion, with the analysis proposed in this paper, we intend to shed some light on the economic meaning of the assumptions that have been used by Kehoe[9] for the extension of the index theorem to constant returns production economies. Our analysis suggests that the results gained by Kehoe's theorem, which are widely quoted in the works on the uniqueness of equilibrium, must be used very carefully, as many of the production models generally studied by economists do not satisfy the assumptions required for its application.

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[9] Although Kehoe's work is doubtless the most important attempt to build an index theorem for constant returns production economies, we do not intend to deny the possibility that there could be some way of obtaining an index theorem for production economies different from Kehoe's (the author must thank an anonymous referee for a discussion about this possibility). On the contrary, by pointing out the weaknesses in Kehoe's method, we intend to foster other possible ways of studying uniqueness for equilibrium models with production.

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