The integrated wage-commodity sector and the surplus equation

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SUMMARY

1. Introduction

The starting point of this paper is represented by the analysis carried out by Sraffa in the first four chapters of *Production of Commodities by Means of Commodities* (Sraffa, 1960). Taking as given: a) the (gross) quantities of commodities produced in the different industries; b) the technical conditions of production; c) the real wage rate, Sraffa showed that it is possible to determine the relative prices of the goods and the profit rate by solving a system of simultaneous equations.

Once the system is obtained, it can be solved for any level of the real wage rate w – taken between 0 and 1 – so as to determine the corresponding levels of the rate profit r. In so doing, we get a relation between w and r. Through the construction of the "standard commodity", Sraffa proved that this relation has the following mathematical form:

$$r = R\,(1 - w) \tag{1}$$

in which R is the "standard ratio", namely the ratio between net product and means of production in the standard system.

The equation that expresses the rate of profit as a function of the wage rate has been called by Garegnani "surplus equation". As Sraffa himself made clear, his surplus equation crucially depends on the possibility to conceive the wage rate as an amount of value expressed in terms of the standard commodity. However, it can easily be proved that, from Sraffa's price equations, a surplus equation – generally non-linear – can always be obtained, whatever the numéraire commodity in which the wage rate is expressed.

The point that we shall try to examine here is whether it is possible to find a surplus equation for the case in which the wage, instead of being an amount of value, is understood, using the words of Sraffa, "on the same footing as the fuel for the engines and the feed for the cattle" (Sraffa 1960: 9). This possibility has been examined by Garegnani is a number of papers – cf. in particular Garegnani (1984, 1987).[1] The present paper is largely based on Garegnani's contributions.

[1] See also Steedman (1977) and Fratini (2015).

2. The model

Following Sraffa, as far as possible, we refer to an economy in which there are K single-product industries, whose production is represented in the usual way:[2]

$$\begin{gathered}
A_a \oplus B_a \oplus \cdots \oplus K_a \oplus L_a \rightarrow A \\
A_b \oplus B_b \oplus \cdots \oplus K_b \oplus L_b \rightarrow B \\
\vdots \\
A_k \oplus B_k \oplus \cdots \oplus K_k \oplus L_k \rightarrow K
\end{gathered}$$

As is well-known, from the system of production it is possible to get the unit coefficients representing the technical conditions of production. Specifically, given the technical conditions of production, we have that $x_y = X_y/Y$ is the quantity of (X) used in order to produce one unit of commodity (Y), with $X = A, B, \ldots, K, L$ and $Y = A, B, \ldots, K$. Hence:

$$\begin{gathered}
a_a \oplus b_a \oplus \cdots \oplus k_a \oplus \ell_a \rightarrow 1 \text{ unit of (A)} \\
a_b \oplus b_b \oplus \cdots \oplus k_b \oplus \ell_b \rightarrow 1 \text{ unit of (B)} \\
\vdots \\
a_k \oplus b_k \oplus \cdots \oplus k_k \oplus \ell_k \rightarrow 1 \text{ unit of (K)}
\end{gathered}$$

The unit coefficients can be organized into a K × K matrix **M** – whose rows refer to commodity and columns to industries – and a 1 × K vector $\boldsymbol{\ell}$ – whose entries are the employments of labour per unit of output in the K industries.

$$\mathbf{M} \equiv \begin{bmatrix} a_a & a_b & \cdots & a_k \\ b_a & b_b & & b_k \\ \vdots & & \ddots & \vdots \\ k_a & k_b & \cdots & k_k \end{bmatrix} \tag{2}$$

$$\boldsymbol{\ell} \equiv [\ell_a \quad \ell_b \quad \cdots \quad \ell_k] \tag{3}$$

[2] In this paper we shall always assume that the production cycle has the same duration in each industry and that this is one year.

As far as the wage rate is concerned, it is assumed here that it is given in physical terms. Accordingly, w is not understood here as an amount of value, but rather as a quantity of a particular composite commodity: the "wage commodity". This way of considering the wage rate represents the main difference between the analysis that follows and the one presented by Sraffa in *Production of Commodities by Means of Commodities*, where w is instead an amount of value expressed in terms of a numéraire.

Let us imagine that through the use of statistical tools it is possible to calculate the quantities of commodities consumed by the households of workers and employees of the economy that we are considering. It is a vector of quantities $\mathbf{\Lambda} = (\Lambda_a, \Lambda_b, \dots, \Lambda_k) \in \mathbb{R}_+^K$. If a quantity of labour L is employed in the economy as a whole, then we can imagine that there is a wage rate w and a vector of quantities $\boldsymbol{\lambda} = (\lambda_a, \lambda_b, \dots, \lambda_k)$ such that: $L \cdot w \cdot \boldsymbol{\lambda} = \mathbf{\Lambda}$.

In so doing, we can say that: i) $\boldsymbol{\lambda}$ is the bundle of commodities that forms one unit of the "wage commodity"; ii) w is the wage rate expressed as a quantity of the "wage commodity"; iii) $w \cdot \boldsymbol{\lambda} = (w \cdot \lambda_a, w \cdot \lambda_b, \dots, w \cdot \lambda_k)$ is the physical wage rate, namely as a bundle of commodities.

3. The system of price equations

For any (row) vector of prices of the K commodities $\mathbf{p} = (p_a, p_b, \dots, p_k) \in \mathbb{R}_+^K$, the value of one unit of the wage commodity can be determined in the following way:

$$p_\lambda = \mathbf{p} \cdot \boldsymbol{\lambda} = p_a \cdot \lambda_a + p_b \cdot \lambda_b + \cdots + p_k \cdot \lambda_k \tag{4}$$

Once the price p_λ is defined, the wage rate in value terms is $w \cdot p_\lambda$. This is the way in which the wage rate must be included among the costs in order to write the price equations:

$$\begin{cases} p_a \cdot A = (1+r)(p_a \cdot A_a + p_b \cdot B_a + \cdots + p_k \cdot K_a) + w \cdot p_\lambda \cdot L_a \\ p_b \cdot B = (1+r)(p_a \cdot A_b + p_b \cdot B_b + \cdots + p_k \cdot K_b) + w \cdot p_\lambda \cdot L_b \\ \vdots \\ p_a \cdot K = (1+r)(p_a \cdot A_k + p_b \cdot B_k + \cdots + p_k \cdot K_k) + w \cdot p_\lambda \cdot L_k \end{cases} \tag{5}$$

The meaning of the equations is exactly the same as in Sraffa's book: in each sector, the amount of revenues is equal to the amount costs plus the normal profit on the capital invested.[3] The only difference with Sraffa's equation concerns the meaning of the variable w. It is an amount of value in Sraffa's equation and a quantity of the wage commodity in system (5). Being a quantity, in system (5) w is multiplied by the corresponding price p_λ. System (5) has, therefore, an unknown more than Sraffa's one. In fact, considering w as an exogenous variable, there are K + 2 unknowns to determine: K + 1 prices and the rate of profit r. Accordingly, equation (4) must be included in the system in order to get a solution.

Finally, a numéraire must be adopted as a unit of measure of commodity values. It can be, as is known, any commodity, single or composite. However, for the analysis that we shall carry out in these pages, it is useful to express the value of commodities in Adam Smith's way, namely in terms of "labour commanded". Since the value of a commodity in terms of labour commanded is equal to the quantity of labour that can be purchased (or hired) by selling a unit of that commodity, the value of w unit of commodity wage in terms of labour commanded is, by definition, equal to 1. That is:

$$w \cdot p_\lambda = 1 \tag{6}$$

Equations (4), (5) and (6) form a system of K + 2 equations with K + 2 unknowns to determine.

Clearly, the system can be simplified and re-written in an equivalent form. In particular, we can start by dividing each equation of system (5) by the quantity produced of the commodity to whose sector it refers. Besides, we can use equation (6) and replace $w \cdot p_\lambda$ with 1. In so doing we obtain the following equations:

$$\begin{cases} p_a = (1+r)(p_a \cdot a_a + p_b \cdot b_a + \cdots + p_k \cdot k_a) + \ell_a \\ p_b = (1+r)(p_a \cdot a_b + p_b \cdot b_b + \cdots + p_k \cdot k_b) + \ell_b \\ \vdots \\ p_k = (1+r)(p_a \cdot a_k + p_b \cdot b_k + \cdots + p_k \cdot k_k) + \ell_k \end{cases} \tag{7}$$

that is, in a more compact form:

[3] As in Sraffa, wages are assumed to be paid *post factum*, hence they are not advanced from capital but paid directly from the revenues. Accordingly, the investment of capital corresponds to the value of the means of production.

(7′) $$\mathbf{p} = (1 + r) \cdot \mathbf{p} \cdot \mathbf{M} + \boldsymbol{\ell}$$

And, as for the numéraire equation:

(8) $$w \cdot \mathbf{p} \cdot \boldsymbol{\lambda} = 1$$

Equations (7′) and (8) form the system whose solution determine the vector of commodity prices **p** – in terms of labour commanded – and the rate of profit *r*.

4. The integrated wage-commodity sector (IWCS)

If we take any industry, for example the industry of commodity (A), we have that the net product of this industry can be calculated only in value terms, as the difference between the value of the output and that of the means of production. It cannot be determined in physical terms because the output is made up of the commodity (A) only, while the means of production generally include also positive quantities of the other commodities. However, if we took a group of industries, suitably sized, we could build an "integrated sector" capable of giving a net physical output of commodity (A). This integrated sector of commodity (A), therefore, would give a gross output formed by a certain quantity of commodity (A) – which is the final output – plus the quantities of commodities (A), (B), ..., (K) that the same sector employs as means of production in a cycle. Accordingly, the only input that this integrated sector would take from outside is labour.

Similarly, the integrated wage-commodity sector (IWCS, hereafter) is made up of a set of industries, taken in such proportions that the net product of the integrated sector consists of the composite wage commodity. In particular, if the quantity of labour employed in the whole economy is equal to L units – i.e. $L = L_a + L_b + \cdots + L_k$, then it is possible to build an integrated sector whose physical net output corresponds to the commodities that allows the payment of the physical wages for L units of labour. In other words, the physical net product of the IWCS is: $NP_w = L \cdot w \cdot \boldsymbol{\lambda} = \boldsymbol{\Lambda}$.

In order to build this integrated sector, we need, first of all, to determine which gross production of the K industries allows us to obtain a physical net product equal to the vector $\mathbf{\Lambda}$. Let $\mathbf{Q} \in \mathbb{R}_+^K$ be the gross output of the IWCS, the vector of commodities used as means of production in this sector is $\mathbf{M} \cdot \mathbf{Q}$. Since, by definition, the gross product is equal to the net product plus the means of production – namely: $GP_w = NP_w + MP_w$ – then:

(9) $$\mathbf{Q} = \mathbf{\Lambda} + \mathbf{M} \cdot \mathbf{Q}$$

We have therefore a linear system with K equations and unknowns: the quantities in the vector $\mathbf{Q}$. Let $\mathbf{I}$ be the K × K identity matrix,[4] the solution of the system is:

(10) $$GP_w = \mathbf{Q} = (\mathbf{I} - \mathbf{M})^{-1} \cdot \mathbf{\Lambda}$$

Once the vector of commodities that form the gross output of the IWCS is determined, the requirements of means of production and labour in this sector can easily be calculated. Specifically:

(11) $$MP_w = \mathbf{M} \cdot \mathbf{Q}$$

(12) $$L_w = \boldsymbol{\ell} \cdot \mathbf{Q}$$

BOX

Adopting an expanded notation, let $(A_w, B_w, \cdots, K_w) \in \mathbb{R}_+^K$ be the vector of commodities employed as means of production in the IWCS, then:

$$\begin{gathered} A_w = a_a \cdot Q_a + a_b \cdot Q_b + \cdots + a_k \cdot Q_k \\ B_w = b_a \cdot Q_a + b_b \cdot Q_b + \cdots + b_k \cdot Q_k \\ \vdots \\ K_w = k_a \cdot Q_a + k_b \cdot Q_b + \cdots + k_k \cdot Q_k \end{gathered}$$

And similarly, as far as labour is concerned:

[4] An identity matrix is a square matrix with 1s on the main diagonal and 0s elsewhere. Let $\mathbf{v}$ be a n × 1 vector and $\mathbf{I}$ a n × n identity matrix, then $\mathbf{I} \cdot \mathbf{v} = \mathbf{v}$.

$$L_w = \ell_a \cdot Q_a + \ell_b \cdot Q_b + \cdots + \ell_k \cdot Q_k$$

Summing up, in the IWCS, in every production cycle, a quantity of labour L_w is employed beside a vector of means of production MP_w with the aim of producing a vector of gross outputs GP_w. The difference between GP_w and MP_w – i.e. the vector of net outputs NP_w – is exactly the vector of commodities that are needed in order to pay the physical wages for the total amount of labour employed in the economy as a whole.

5. Profit per unit of labour in the IWCS

Our interest in the IWCS stems from two of its characteristics.

I) In this sector, net product and wages are physically made up of the same commodity: the wage commodity. Consequently, the amount of profit earned by capitalists in this sector can be determined independently of prices, either as a vector of commodities, or as a quantity of the wage commodity:

(13) $$\mathbf{\Pi}_w = \mathbf{\Lambda} - L_w \cdot w \cdot \boldsymbol{\lambda} = (L - L_w) \cdot w \cdot \boldsymbol{\lambda}$$

Therefore, the amount of profit of the IWCS is $(L - L_w) \cdot w$ units of the wage commodity.

II) Adopting labour commanded as the unit of measure of commodity values – and hence positing $w \cdot \mathbf{p} \cdot \boldsymbol{\lambda} = 1$, cf. equation (8) – the amount of profit realized in the IWCS, expressed in value terms, turns out equal to the difference between the amount of labour L employed in the whole economy and that employed in the sector L_w. In fact:

(14) $$\Pi_w = \mathbf{p} \cdot \mathbf{\Pi}_w = (L - L_w) \cdot w \cdot \mathbf{p} \cdot \boldsymbol{\lambda} = L - L_w$$

Hence, the amount of profit earned by capitalists in the IWCS, in value terms, does not depend on commodity relative prices and, accordingly, can be determined before these prices are calculated.

However, the amount of profit Π_w still depends on the dimension of the economy we are dealing with. In fact, for instance, if the economy would grow by 50%, other things being equal, then the employment of labour would also grow by the same rate and thus the wages to be paid. Accordingly, the employment of labour in the IWCS would also have to grow by 50%. Therefore, in the new, increased economy, the profit in the IWCS would be: $1.5\,L - 1.5\,L_w = 1.5(L - L_w)$. Namely, it would be equal to the initial profit increased by 50%. We can, however, disregard this scale factor by referring, in our analysis, to the amount of profit per unit of labour:

$$\pi_w = \frac{\Pi_w}{L_w} = \frac{L - L_w}{L_w} = \frac{L}{L_w} - 1 \tag{15}$$

The ratio $(L - L_w)/L_w$ presents some analogies with the rate of surplus-value used by Marx in his analysis of the rate of profit. Marx denotes by C, V and S the constant capital, the variable capital[5] and the surplus-value, respectively, expressed in terms of labour embodied. In his analysis, the general rate of profit is $r = S/(C+V) = (S/V)/(C/V+1)$. The ratio C/V is called "organic composition of capital" and S/V is the "rate of surplus-value" or "rate of exploitation". Both the ratio $(L - L_w)/L_w$ and the rate of surplus value S/V are the amount of profit divided by the amount of wages. There are, however, two differences: i) Marx's ratio S/V refers to the entire economy while $(L - L_w)/L_w$ refer to the IWCS only; ii) S and V are expressed in terms of labour embodied while $(L - L_w)$ and L_w are amounts of labour commanded.[6]

[5] In Marx's theory, as is known, the constant capital is the value of the means of production and the variable capital is the value of the wages (paid in advance).

[6] It can also be stressed that, numerically $S/V = (L - L_w)/L_w$. In fact, the amount of labour employed in the IWCS is clearly the amount of labour embodied into the commodities given to workers, hence $V = L_w$. Moreover, L –i.e. the total employment – is the quantity of labour embodied in the national net output, and, therefore, $L - V = L - L_w$ is the surplus-value in terms of labour embodied S.

6. Capital per unit of labour in the IWCS

Having obtained the profit per unit of labour π_w, if we also knew the value of the capital per unit of labour in the IWCS – denoted by v_w, we could determine the profit rate *r* as the ratio between the two: $r = \pi_w / v_w$. The problem is that while the profit per unit of labour π_w, expressed in labour commanded, has been determined simply on the basis of the employments of labour L and L_w, the value of the capital per unit of labour v_w, although expressed in labour commanded, depends on commodity relative prices.

With equations (11) and (12), we have seen that the employments of means of production and labour in the IWCS are $MP_w = \mathbf{M} \cdot \mathbf{Q}$ and $L_w = \boldsymbol{\ell} \cdot \mathbf{Q}$ respectively. Accordingly, the employment of means of production per unit of labour is a vector $\boldsymbol{\mu} = (\mu_a, \mu_b, \cdots, \mu_k) \in \mathbb{R}_+^K$ such that:

$$\boldsymbol{\mu} = \frac{1}{L_w} \cdot \mathbf{M} \cdot \mathbf{Q} \tag{16}$$

It has already been said that, in the economy we are considering, since wages are paid *post factum*, the costs anticipated by capital correspond to the value of the means of production. Consequently, with regard to the investment of capital per unit of labour in the IWCS, we have that:

$$v_w = \mathbf{p} \cdot \boldsymbol{\mu} = p_a \cdot \mu_a + p_b \cdot \mu_b + \cdots + p_k \cdot \mu_k \tag{17}$$

It is therefore clear that commodity prices are needed in order to determine the investment of capital per unit of labour in the IWCS. Now, these prices could be determined, as we have seen in section 3, solving the system formed by the equations (7') and (8). However, these equations would determine, simultaneously with the prices, also the rate of profit *r*, which therefore would be already fixed. By contrast, here we want to build a surplus equation and, accordingly, the rate of profit must be a variable. For this reason, while starting from the price equations, in the next paragraph, instead of solving them, we will use them to express prices as a function of the profit rate.

7. The "reduction" of commodity prices to "dated quantities of labour"

In chapter VI of *Production of Commodities by Means of Commodities*, Sraffa calls "reduction to dated quantities of labour" the operation by which, in the price equations, the means of production are replaced by a series of quantities of labour performed in previous periods in order to obtain those means of production. In this way, while the quantities of the means of production are replaced by the quantities of labour employed in the past, their value is replaced by the wages anticipated in previous periods, multiplied by the appropriate factor of profit. This, as we will see shortly, makes prices disappear on the RHS of the equations, making the values of commodities in labour commanded dependent exclusively on the technical conditions of production and the profit rate.

Let us start from the system of equations (7), already written in section 3:

$$(7) \qquad \begin{cases} p_a = (1+r)(p_a \cdot a_a + p_b \cdot b_a + \cdots + p_k \cdot k_a) + \ell_a \\ p_b = (1+r)(p_a \cdot a_b + p_b \cdot b_b + \cdots + p_k \cdot k_b) + \ell_b \\ \vdots \\ p_k = (1+r)(p_a \cdot a_k + p_b \cdot b_k + \cdots + p_k \cdot k_k) + \ell_k \end{cases}$$

Following Sraffa's procedure – which is equivalent, in fact, to solving the system – commodity prices can be expressed as the sum of the wages for current labour (per unit of output) plus the wages for the series of dated labour (per unit of output as well), suitably multiplied by a profit factor.

Using Sraffa's notation, we denote by ℓ_x the unit coefficient of direct labour used in the production of commodity (X), with X = A, B, ..., K; by ℓ_{x1} the unit coefficient of labour employed one period before; by ℓ_{x2} the unit coefficient of two-period backdated labour; ... by ℓ_{xt} the unit coefficient of t-period backdated labour. In so doing, we have that $\ell_x + \sum_{t=1}^{\infty} \ell_{xt}$ is the quantity of labour that one unit of commodity (X) embodies, and $p_x = \ell_x + \sum_{t=1}^{\infty} \ell_{xt}\,(1+r)^t$ is the price of one unit of commodity (X) in terms of labour commanded. This price can be seen as a function of the rate of profit: $p_x(r) \equiv \ell_x + \sum_{t=1}^{\infty} \ell_{xt}\,(1+r)^t$. Hence:

(18) $$\begin{cases} p_a(r) \equiv \ell_a + \sum_{t=1}^{\infty} \ell_{at}\,(1+r)^t \\ p_b(r) \equiv \ell_b + \sum_{t=1}^{\infty} \ell_{bt}\,(1+r)^t \\ \vdots \\ p_a(r) \equiv \ell_k + \sum_{t=1}^{\infty} \ell_{kt}\,(1+r)^t \end{cases}$$

If production is "non-circular" – i.e. there is at least one commodity produced only with labour (without means of production) and no commodity is (directly or indirectly) a means of production of itself – then any commodity can be obtained without the need of an initial endowment of means of production, provided that the beginning of the process occurred a reasonable number of periods before.

If instead the production is "circular", as it is in general, it will not be possible to complete the reduction with a finished number of terms of labour, but there will always be, as Sraffa writes, a "commodity residue". In other words, the beginning of the process, even if backdated at will, will always require, in addition to the labour, the means of production. In this case, $\ell_x + \sum_{t=1}^{\infty} \ell_{xt}\,(1+r)^t$ is the sum of infinite addends. However, it is possible to prove – under the assumption that the economy is technically able to produce a surplus – that this sum converges to a finite value for each level of the profit rate *r* below a certain maximum.

BOX

As is well-known, the reduction of prices to dated quantities of labour is equivalent to the solution of the system, with r considered as an independent variable. In fact, from equation (7′) – i.e. system (7) re-written in a vectorial form – we get:[7]

(19) $$\mathbf{p} = \boldsymbol{\ell} \cdot [\mathbf{I} - (1+r)\cdot \mathbf{M}]^{-1} \equiv \mathbf{p}(r)$$

Now, under some weak restrictions about matrix **M**,[8] we have that:

(20) $$[\mathbf{I} - (\mathbf{1}+\mathbf{r})\cdot \mathbf{M}]^{-1} = \mathbf{I} + (1+r)\cdot \mathbf{M} + [(1+r)\cdot \mathbf{M}]^2 + \cdots =$$
$$= \mathbf{I} + \sum_{\boldsymbol{t}=\mathbf{1}}^{\infty} [(1+r)\cdot \mathbf{M}]^t$$

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[7] Equation (7′) is: $\mathbf{p} = (1+r)\cdot \mathbf{p}\cdot \mathbf{M} + \boldsymbol{\ell}$; therefore: $\mathbf{p}\cdot[\mathbf{I} - (1+r)\cdot \mathbf{M}] = \boldsymbol{\ell}$. If the matrix $[\mathbf{I} - (1+r)\cdot \mathbf{M}]$ is invertible, then we get equation (19).

[8] See Seedman (1977: 69-76), Pasinetti (1977: 89-92) and Kurz and Salvadori (1995: 165-168) for details.

Therefore:

(18′) $$\mathbf{p}(r) \equiv \boldsymbol{\ell} + \boldsymbol{\ell} \cdot \sum_{\boldsymbol{t=1}}^{\infty} [(1+r) \cdot \mathbf{M}]^{t}$$

Equations (18′) is nothing else than the set of equations (18). Therefore, if matrix M satisfies the required conditions, the equality in equation (20) proves that, for each commodity (X), with X = A, B, …, K, the sum $\ell_x + \sum_{t=1}^{\infty} \ell_{xt}\,(1+r)^t$ converges to a finite value.

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8. The profit function

We have just seen that the price of any commodity – through the "reduction" to wages for dated employments of labour – can be expressed as a function of the profit rate: $\mathbf{p} = \mathbf{p}(r)$. We can therefore incorporate this result into the reasoning in section 6. Let us start, in particular, from equation (17), which defines the value, in terms of labour commanded, of the means of production employed in the IWCS per unit of labour. In that equation, commodity price can be replaced with the corresponding function $\mathbf{p} = \mathbf{p}(r)$.

(21) $$v_w(r) \equiv \mathbf{p}(r) \cdot \boldsymbol{\mu} \equiv \boldsymbol{\ell} \cdot [\mathbf{I} - (1+r) \cdot \mathbf{M}]^{-1} \cdot \boldsymbol{\mu}$$

Or, which is the same:

(21’) $$v_w(r) \equiv \sum_{x=a}^{k} p_x(r) \cdot \mu_x = \sum_{x=a}^{k} [\ell_x + \sum_{t=1}^{\infty} \ell_{xt}\,(1+r)^t] \cdot \mu_x =$$
$$= \sum_{x=a}^{k} \ell_x \cdot \mu_x + \sum_{t=1}^{\infty} [\sum_{x=a}^{k} \ell_{xt} \cdot \mu_x] \cdot (1+r)^t$$

Consequently, being the prices of the functions of the profit rate *r*, also the value of the means of production is now a function of *r*: $v_w = v_w(r)$. We now have all the elements to write the surplus equation. However, instead of writing the equation in the most natural form – i.e. $r = \pi_w / v_w$, it is convenient, for the reasons that we will clarify shortly, to write it in the form $\pi_w = f(r)$. This function, called "profit function" by

Garegnani (1984: 317), expresses the amount of profit per worker that is needed in order to remunerate the capital invested in the IWCS at a rate r:

$$(22) \qquad \pi_w = f(r) \equiv r \cdot v_w(r) \equiv r \cdot \{\sum_{x=a}^{k} \ell_x \cdot \mu_x + \sum_{t=1}^{\infty} [\sum_{x=a}^{k} \ell_{xt} \cdot \mu_x] \cdot (1+r)^t\}$$

In other words, if the rate of profit is r, an investment of capital $v_w(r)$ yields an amount of profit $f(r)$.

As for the shape of the function $\pi_w = f(r)$, the following remarks can be made.

Remark 1: $f(0) = 0$.

Remark 2: $f(r)$ is monotonically increasing – i.e. $f'(r) > 0$. In fact, as we can see by the equation (22), the profit rate r is always elevated to positive powers (precisely equal to or greater than 1) and multiplied by non-negative coefficients, since ℓ_x, ℓ_{xt} and μ_x are surely non-negative quantities ($\forall$ $x = a, b, \ldots, k$ and $\forall$ $t = 1, 2, \ldots$).

Remark 3: $f(r)$ is convex – i.e. $f''(r) > 0$. This follows from remark 2, noting that r is always elevated to powers equal to or greater than 1.

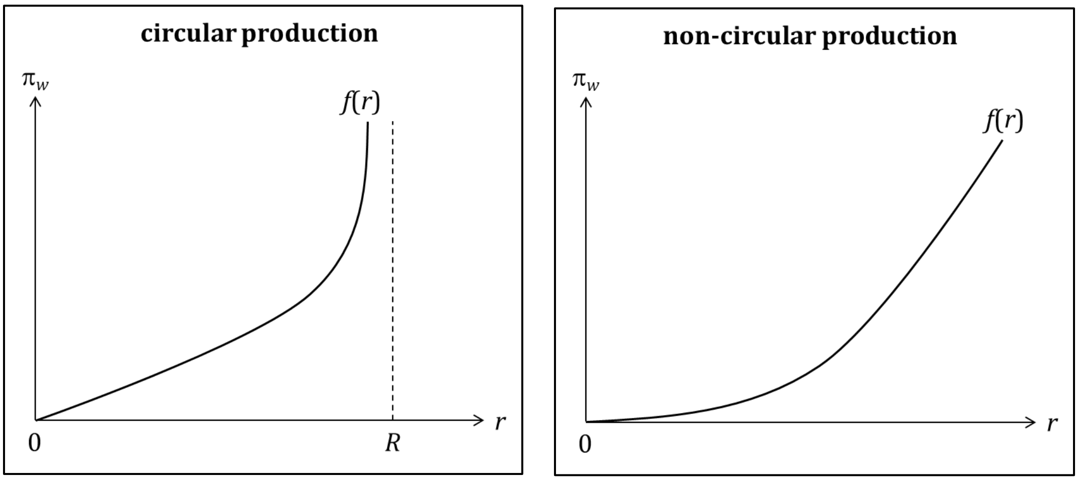
Remark 4: If production is non-circular, then $f(r)$ is a finite magnitude for every finite level of r.
If the production is non-circular, then the price of any commodity can be completely reduced to a finished number of dated quantities of labour. This makes $f(r)$ a polynomial function of grade T+1, where T is the maximum number of labour terms at which commodity prices are reduced.

Remark 4: If production is circular, then $\exists\ R \in \mathbb{R}_+ : r \rightarrow R \Rightarrow f(r) \rightarrow \infty$.
This will be demonstrated intuitively. It has been said that if production is circular, then at least one commodity enters directly or indirectly into its own production process. Let us, therefore, consider the simplest case: that of a commodity produced with only labour and itself. The price equation of this commodity, in labour commanded, will be: $p_x = x_x \cdot p_x \cdot (1+r) + \ell_x$. It can be rewritten, with the appropriate steps, as follows: $\ell_x/p_x = 1 - x_x - x_x \cdot r$. Consequently, when $r = (1-x_x)/x_x$, also called "reproduction rate" of

commodity (X), the RHS of the equation becomes nil, so the LHS must fall to zero as well. This means that $p_x \to \infty$ as $r \to (1 - x_x)/x_x$. In other words, as $r \to (1 - x_x)/x_x$, the value of capital employed $x_x \cdot p_x$ (in terms of labour commanded) tends to infinity and, accordingly, the amount of profit needed to remunerate this capital at a rate *r* tends to infinity as well.

Therefore, in the case of circular production, when $r \to R$, the price in labour commanded of one or more means of production tends to infinity, making infinity also v_w and, consequently, the profits necessary to remunerate this capital at a rate *R*.

Fig. 1 – The profit function



9. The surplus equation

Summing up, we have first calculated the profit per unit of labour in the IWCS, in labour commanded:

(15’)
$$\pi_w = \frac{L - L_w}{L_w}$$

Then we have built the profit function:

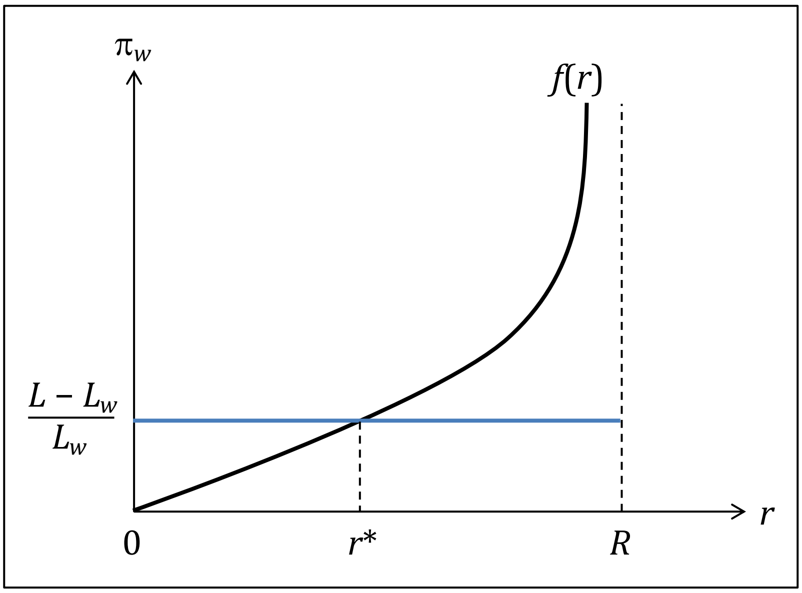
(22’)
$$\pi_w = f(r)$$

Now, we can put these two results together into a single equation, the surplus equation:

(23) $$\frac{L-L_w}{L_w} = f(r)$$

The solution of equation (23) can be obtained graphically, representing equations (15') and (22') in the same system of coordinate axes, in which the rate of profit is on the horizontal axis and the amount of profit per unit of labour is on the vertical axis. We already know all the properties of the function $\pi_w = f(r)$. As far as the function $\pi_w = (L - L_w)/L_w$ is concerned, being independent of r, it is a horizontal line.

Fig. 2 – The surplus equation



Since we know that $f(r)$ is a monotonically increasing function that starts from zero and tends to infinity, it surely intersects the horizontal line. This intersection determines the rate of profit r^*, as in fig. 2. This is the rate of profit that can be paid on the capital invested in the IWCS with an amount of profit per unit of labour equal to $(L - L_w)/L_w$.

Compared to the method of price equations, the surplus equation has the advantage of better highlighting the inverse relationship between the profit rate and the wage rate. In particular, we can consider here the case of a decrease – say of a certain percentage δ, with $0 < \delta < 1$ – in the amount of wage commodity given to workers per unit of labour. In other words, let us suppose that now the wage rate has become w', with $w' = w \cdot (1 - \delta)$, while the physical composition of the wage commodity $\boldsymbol{\lambda} = (\lambda_a, \lambda_b, \dots, \lambda_k)$ has remained unchanged.

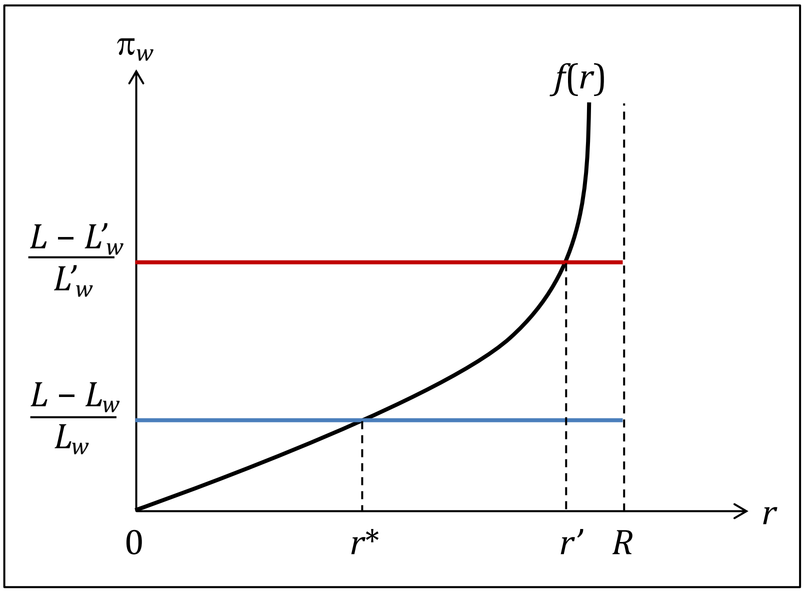
As a result of the reduction in the wage rate in physical terms, the physical net product of the IWCS will now also be reduced. In fact, we will have $L \cdot w' \cdot \boldsymbol{\lambda} = \boldsymbol{\Lambda}'$ and hence $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda} \cdot (1 - \delta)$.

Since, in the IWCS, the net output falls, the gross output falls as well: $\mathbf{Q}' = [\mathbf{I} - \mathbf{M}]^{-1} \cdot \boldsymbol{\Lambda}'$ implies $\mathbf{Q}' = \mathbf{Q} \cdot (1 - \delta)$. Accordingly, a smaller amount of labour is now employed in the IWCS: $L'_w = \boldsymbol{\ell} \cdot \mathbf{Q}'$ and $\mathbf{Q}' = \mathbf{Q} \cdot (1 - \delta)$ imply $L'_w = L_w \cdot (1 - \delta)$.

Conversely, the value of the net product of the IWCS in terms of labour commanded labour has not changed. The IWCS is built in such a way that its net physical net product is formed precisely by the commodities used to pay the wages for the total labour force of the economy L. Hence, $\mathbf{p} \cdot \boldsymbol{\Lambda}'$ is L as before, because now $w' \cdot \mathbf{p} \cdot \boldsymbol{\lambda} = 1$.

Consequently, the decrease in the physical wage rate leads to an increase in the profit per unit of labour, since: $L/L'_w > L/L_w$. So, in our figure, the horizontal line representing the profit per unit of labour in the IWCS is shifted upwards.

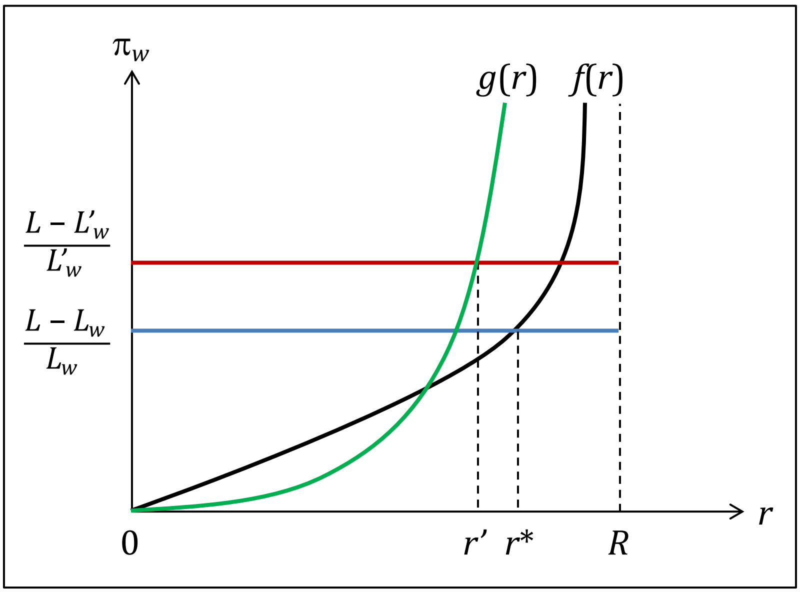
Fig. 3 – The surplus equation with a lower wage rate



The profit function, conversely, does not change as a result of the change in the wage rate, provided that the physical composition of the wage commodity and the technical conditions of production remain unchanged. In fact, on the one hand, since the value of the wages rate remained equal to 1 and since the technical coefficients of production remained unchanged, the reduction of prices to wages for dated quantities of labour – represented by the vector of functions $\mathbf{p}(r)$ – remained that determined by the system (18). On the other hand, the means of production per worker also remained unchanged, as $\mathbf{M} \cdot \mathbf{Q}/L_w = \mathbf{M} \cdot \mathbf{Q}'/L'_w$.

It is concluded that the decrease in the physical wage rate – for the same wage commodity – causes an increase in the profit per unit of labour, which generates an increase in the profit rate, from r^* to r' (see Fig. 3). The inverse relationship between the two distribution variables is therefore further confirmed.

Fig. 4 – Change in the wage commodity



If, conversely, the variation of the wage rate did not occur with the same physical composition of the wage commodity, then the analysis would become much more complicated. In fact, if the physical composition of the wage commodity were to change, the levels of the wage rate before and after the change would be incomparable. In this case, w and w' would be the quantities of two different (composite) commodities and therefore, even if $w > w'$, one could not say either that the physical wage rate has decreased or that it has increased. We can get a better information looking at the employments of labour in the IWCS. For instance, we can have $L_w > L'_w$ and this is in some sense equivalent to a reduction of the wage rate. However, when the physical composition of the wage commodity changes, the profit function changes as well and so we are not guaranteed that the fall in the employment of labour in the IWCS will bring about an increase in the rate of profit.

This possibility is considered in fig. 4, where a decrease in the employment of labour in the IWCS is associated with a decrease in the rate of profit because of the shape of the new profit function $g(r)$.

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