Existence and uniqueness of the solution of Sraffa's first equations

Saverio M. Fratini

April 2019

Teaching Material n.1

Existence and uniqueness of the solution of Sraffa's first equations. An example with three commodities

We consider a system of production for subsistence with three commodities. They are denoted as "a", "b" and "c"—instead of "wheat", "iron" and "pigs", as in Sraffa's book (1960, p. 4).

System of production

$$A_a \oplus B_a \oplus C_a \rightarrow A$$

$$A_b \oplus B_b \oplus C_b \rightarrow B$$

$$A_c \oplus B_c \oplus C_c \rightarrow C$$

As in Sraffa, $X_y \geq 0$ is the quantity of commodity "x" employed in the production of commodity "y" and $Y > 0$ is the quantity of commodity "y" produced (with "x" and "y" = "a", "b", "c").

For this system of production, commodity relative prices are determined as the solution of the following system of price equations:

$$A_a p_a + B_a p_b + C_a p_c = A p_a$$
$$A_b p_a + B_b p_b + C_b p_c = B p_b$$
$$A_c p_a + B_c p_b + C_c p_c = C p_c$$

Assumptions

A1. The system is in a self-replacing state. That is to say: $A_a + A_b + A_c = A$; $B_a + B_b + B_c = B$; $C_a + C_b + C_c = C$.

A2. Commodities "a", "b" and "c" are "basic commodities", i.e. each of them enters directly or indirectly into the production of all the commodities.

Proposition

If assumptions A.1 and A.2 hold, then the system of price equations determines one and only one set of strictly positive relative prices.

Proof

Let us start by re-writing the system of price equations:

$$(A - A_a)p_a - B_a p_b - C_a p_c = 0$$
$$-A_b p_a + (B - B_b)p_b - C_b p_c = 0$$
$$-A_c p_a - B_c p_b + (C - C_c)p_c = 0$$

Now, it is clear that the system is a homogeneous linear system, whose trivial solution is $p_a = p_b = p_c = 0$. Therefore, in order to study the possibility of non-trivial solutions, we need to study the properties of the coefficient matrix:

$$\mathbf{M} = \begin{bmatrix} A - A_a & -B_a & -C_a \\ -A_b & B - B_b & -C_b \\ -A_c & -B_c & C - C_c \end{bmatrix}$$

Let us denote by $\mathbf{m}_1$ the first row of matrix M; by $\mathbf{m}_2$ the second row; and by $\mathbf{m}_3$ the third. Assumption A.1 implies that:

$$\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 = \mathbf{0}$$

where $\mathbf{0}$ is the null vector in $\mathbb{R}^3$. Accordingly, matrix $\mathbf{M}$ does not have full rank[1] and, therefore, non-trivial solutions exist.

Moreover, we can prove that matrix $\mathbf{M}$ is of rank two. In order to do that, we begin by proving that there are not two scalars α and β such that $\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 = \mathbf{0}$. In fact, if α and β have opposite signs, then $\alpha(A - A_a) + \beta(-A_b) \neq 0$, because $(A - A_a) > 0$ (assumption A.2) and $(-A_b) \leq 0$. If, instead, α and β have the same sign, then $\alpha(-C_a) +$

[1] The rank of a matrix corresponds to the maximal number of linearly independent rows or columns of a matrix.

$\beta(-C_b) \neq 0$, because $(-C_a) \leq 0$ and $(-C_b) \leq 0$, but at least one of them must be surely negative due to assumption A.2. In both the cases we cannot obtain the null vector as a linear combination of the first two rows of matrix $\mathbf{M}$. With analogous argument, it can be proved that however we take two rows of matrix $\mathbf{M}$, their linear combination cannot be the null vector.

Therefore, all the non-trivial solutions of the system are on the same ray. More precisely, let $\mathbf{p} = [p_a, p_b, p_c]$ be a non-trivial solution of the system, then $\mathbf{p}' = [p_a', p_b', p_c']$ is a solution of the system if and only if there is a scalar λ such that $\mathbf{p} = \lambda\mathbf{p}'$. This means that the relative prices p_b/p_a and p_c/p_a are univocally determined.[2] Having established this, for the non-trivial solution of system to be economically meaningful, the relative prices must all be strictly positive.

The relative prices p_b/p_a and p_c/p_a are strictly positive if the prices p_a, p_b and p_c are either all strictly positive, or all negative. In other words, the non-trivial solution of system would not be economically meaningful if some prices are strictly positive and some other negative. In the latter case there would certainly be negative relative prices.

With the aim of proving that the non-trivial solution of the system is economically meaningful, let us start by assuming that the price of commodity "a" is negative, that is, let us assume $p_a < 0$.

From the first equation of the system we have: $(A - A_a)p_a = B_a p_b + C_a p_c$. Since $(A - A_a) > 0$ (assumption A.2), if $p_a < 0$, then the LHS is surely negative and, accordingly, at least one of the other two prices must be negative as well. Let us say it is p_b.

From the sum of the first and the second equation we have: $(A - A_a - A_b)p_a + (B - B_a - B_b)p_b = (C_a + C_b)p_c$. If $p_a < 0$ and $p_b < 0$, then the LHS is surely negative[3] and, accordingly, also the price p_c must be negative.

In conclusion, either the price p_a, p_b and p_c are strictly positive, or they must all be negative. It is not possible for them to have opposing signs. Therefore, the non-trivial solution univocally determines economic meaningful – i.e. strictly positive – relative prices p_b/p_a and p_c/p_a.

QED

[2] In fact: $p_b'/p_a' = \lambda p_b/\lambda p_a = p_b/p_a$ and similarly $p_c'/p_a' = p_c/p_a$.

[3] We know that $(A - A_a - A_b) \geq 0$ and $(B - B_a - B_b) \geq 0$, but at least one of them must be strictly positive because, otherwise, commodities "a" and "b" would not be basic commodities, violating assumption A.2.

References

Maroscia, P. (2008) Some Mathematical Remarks on Sraffa's Chapter I. In: G. Chiodi and L. Ditta (eds) *Sraffa or an Alternative Economics*. Palgrave Macmillan: Basingstoke and New York.

Sraffa, P. (1960) *Production of Commodities by Means of Commodities*. Cambridge University Press: Cambridge.