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# RESWITCHING OF TECHNIQUES IN AN INTERTEMPORAL EQUILIBRIUM MODEL WITH OVERLAPPING GENERATIONS

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We study an overlapping generation model of Diamond's type. In contrast to Diamond, we do not assume that the technology can be represented by an aggregate neoclassical production function. Rather, we study the case in which (i) the technology consists of a finite number of constant coefficient productive activities, (ii) there are capital goods physically heterogeneous with respect to each other and to the economy's only consumption good and (iii) there are two alternative production methods for obtaining the consumption good. We explore the possible linkage between reswitching of techniques and the multiplicity of stationary state equilibria of the model.

JEL Classification: C62, D33, D90.

## I. INTRODUCTION

The overlapping generations model undoubtedly represents one of the most important and fertile attempts to develop a theory of intertemporal equilibrium outside the Arrow–Debreu paradigm. The substantial difference between the Arrow–Debreu and overlapping generations models lies in the number of agents and commodities: finite for Arrow–Debreu and infinite for overlapping generations.

Even though the literature on equilibrium with overlapping generations is vast and several interesting properties of the model have received attention (see Geanakoplos, 1987, for a survey), one important aspect seems to have remained hitherto unexplored: the choice of techniques with heterogeneous capital goods. The present paper's aim is to take a first step towards filling the gap.

Almost 40 years later, Diamond's model of 1965 continues to be the main reference point for studies on intertemporal equilibrium with overlapping generations and production. Yet there are reasons for dissatisfaction with the model, since it requires "capital and output to be the same commodity", and "technology is assumed to

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be representable by a constant returns to scale production function, F(K,L)" (Diamond, 1965, p. 1127).

The model we shall discuss here is identical to Diamond's in its general conception of how the economy works, but differs as regards the way in which the technology of production is represented. In particular, we study the case in which: (i) the technology consists of a finite number of constant coefficient productive activities, (ii) there are capital goods physically heterogeneous with respect to each other and the only consumption good of the economy, and (iii) there are two alternative production methods for obtaining the consumption good ("wheat").

In the light of the capital debates between the two Cambridges, it is known that in the case we consider there is the possibility of a reverse capital deepening generated by the 'reswitching of techniques'. In other words, as the interest rate rises, the production technique with the higher capital intensity may be rejected, then come back into use, generating an increase in capital intensity associated with an increase in the interest rate.

We will show that the reswitching and the reverse capital deepening implied by it, although envisaged initially with reference to traditional versions of neoclassical theory (cf. Garegnani 1990), may also be found in more recent equilibrium models such as those with overlapping generations. In particular, also using a numerical example, we will see how, in these models, the reverse capital deepening may be a further cause of multiple stationary equilibria, in addition to the causes already known and studied in the literature.

In so doing, we face the problem that the overlapping generation model has, in general, multiple equilibria even in the case of pure exchange with identical consumers (cf. Samuelson 1958 and Kehoe 1991). Therefore, the problems arising from reswitching may be difficult to isolate from other possible causes of multiplicity, leading some authors to raise doubts about the relevance of reswitching as a source of problems.[1]

The main aim of the present paper is that of overcoming these difficulties. In particular, we will focus on a kind of multiplicity that, as will be argued, cannot emerge in the pure exchange case, or in the case of a well-behaved mechanism of choice of techniques. In Section VIII, we will also stress some peculiar characteristics of this kind of multiplicity.

## II. THE CONSUMPTION SIDE

Consider an intertemporal model covering an infinite number of periods, each identified by an integer number.[2] In each period, $L$ individuals with a life of two periods are

[1] See, for example, Bloise and Reichlin (2005).

[2] In the economy with overlapping generations considered here, as well as in the model considered in Geanakoplos (1987), "discrete time periods $t$ extend indefinitely into the past and into the future" (p. 208). The period $t = 0$ may be considered as "today", but not as the initial period, because what happened "yesterday" is not arbitrarily given, but could be explained by the model itself.

born. An individual born in period $t$ gets his intertemporal well-being from the quantity of wheat consumed during his first and second periods, indicated respectively as $g_{1t}$ and $g_{2t+1}$.

As regards preferences, we assume they are identical for all individuals and that they can be represented by the usual utility function with constant elasticity of substitution (CES). In particular, indicating the constant elasticity of substitution with $\theta$ and the intertemporal preference rate with $\rho$, the utility function for an individual born in period $t$ will be:

$$u_t(g_{1t}, g_{2t+1}) = \frac{g_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \cdot \frac{g_{2t+1}^{1-\theta}}{1-\theta} \tag{1}$$

The properties of this utility function are well known. It is continuous and, for $\theta > 0$, the marginal substitution rate between current and future consumption falls as $g_{1t}/g_{2t+1}$ increases.

Turning to endowments, at birth each individual has only one unit of labour for the first period. Consequently, indicating with $w_t$, $p_{gt}$ and $p_{g+1}$, respectively, the wage for a unit of labour in period $t$, and the prices for wheat delivered in periods $t$ and $t+1$, all intended as discounted prices in the sense of Debreu,[3] the individual's budget constraint will be:

$$g_{1t} \cdot p_{gt} + g_{2t+1} \cdot p_{gt+1} = w_t \tag{2}$$

Maximizing the utility function under the budget constraint gives the two demand functions for wheat for an individual born in period $t$:

$$g_{1t} = \frac{w_t}{p_{gt}} \cdot \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (p_{gt}/p_{gt+1})^{(1-\theta)/\theta}} = \omega_t \cdot \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} \tag{3}$$

$$g_{2t+1} = \frac{w_t}{p_{gt}} \cdot \frac{(p_{gt}/p_{gt+1})^{1/\theta}}{(1+\rho)^{1/\theta} + (p_{gt}/p_{gt+1})^{(1-\theta)/\theta}} = \omega_t \cdot \frac{(1+r_{t+1})^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} \tag{4}$$

where $\omega_t$ is the wage rate in period $t$, expressed in terms of wheat in the same period, and $r_{t+1}$ is the interest rate on the loan of wheat from period $t$ to period $t+1$.

During his first year of life an individual born in period $t$ receives a wage equivalent to $\omega_t$ units of wheat delivered at time $t$. Of this wheat, $g_{1t}$ units will be consumed directly at time $t$, and the remaining units will be saved for consumption at time $t+1$. So, in period $t$, the worker's savings expressed in wheat delivered in $t$ will be:

$$\omega_t - g_{1t} = \omega_t \cdot \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} = \omega_t \cdot s(r_{t+1}) \tag{5}$$

[3] For example, $w_t$, $p_{gt}$ and $p_{gt+1}$ could be intended as expressed in terms of wheat delivered in period zero.

The savings of a worker born in period $t$ thus depend on his wage in wheat $\omega_t$, and on his average propensity to save, $s(r_{t+1})$. Moreover, it can be easily proved that:

$$\lim_{r_{t+1}\to\infty} s(r_{t+1}) = 1 \tag{6}$$

and if $0 < \theta < 1$ then:

$$\frac{ds(r_{t+1})}{dr_{t+1}} > 0 \tag{7}$$

Therefore, assuming $0 < \theta < 1$, the average propensity to save $s(r_{t+1})$ is a monotonically increasing function of $r_{t+1}$, approaching 1 as $r_{t+1}$ tends to ∞. In this case, we may say that the average propensity to save function is, from the neoclassical point of view, well-behaved.

## III. PRODUCTION

In each period, three types of product can be obtained: wheat, capital good of type [A] and capital good of type [B]. Both types of capital goods are assumed to be circulating.

Capital goods of type [A] and [B] can be produced by the following fixed coefficient methods:[4]

$$a_{a\ell} \text{ workers} \oplus a_{aa} \text{ capital goods [A]} \to 1 \text{ capital good [A]}$$
$$b_{b\ell} \text{ workers} \oplus b_{bb} \text{ capital goods [B]} \to 1 \text{ capital good [B]}$$

Wheat, on the other hand, can be produced by two alternative methods, one for each kind of capital good:

$$a_{g\ell} \text{ workers} \oplus a_{ga} \text{ capital goods [A]} \to 1 \text{ quintal of wheat}$$
$$b_{g\ell} \text{ workers} \oplus b_{gb} \text{ capital goods [B]} \to 1 \text{ quintal of wheat}$$

With reference to the technical coefficients, we assume, firstly, that the production of capital goods is "viable", i.e. $a_{aa} < 1$ and $b_{bb} < 1$.

It is also assumed that $a_{ga}/a_{g\ell} \geq a_{aa}/a_{a\ell}$ and that $b_{gb}/b_{g\ell} \geq b_{bb}/b_{b\ell}$. In economic terms, we assume that, for each technique, the production of the capital good is not more capital intensive than the production of wheat. As will become clearer subsequently, this hypothesis is made in order to avoid the price of each capital good (in terms of wheat) being an increasing function of the interest rate. When this is

[4] The symbol " ⊕ "denotes the technical combination of inputs into the production process. The symbol " → "denotes the result of such a technical combination, i.e. the output.

not the case, as is well known, the value of the capital employed may rise with the interest rate, without any change occurring in physical capital.

As regards the output's delivery date, we will assume that the wheat and capital goods produced during period $t$ are delivered by the end of the same period. Yet, while the wheat produced and delivered in $t$ is assumed to be consumed within the same period, we shall assume that the capital goods can only be put to productive use in the period after the one in which they are produced and delivered. So we suppose that the capital goods delivered in period $t$ can only and exclusively be used in production during period $t+1$.

We shall use $G_t^a$ and $G_t^b$ to indicate the quantities of wheat produced in period $t$ with capital goods [A] and [B] respectively. $A_t$ and $B_t$ will be used for the quantity of capital goods of the two types produced in $t$. Therefore $Q_t = [G_t^a, G_t^b, A_t, B_t] \in R_+^4$ is a vector of activity levels in $t$.

## IV. THE EQUILIBRIUM PATH

Given the demand functions $g_{1t}(.)$ and $g_{2t}(.)$, the number of individuals in each generation $L$, and the technical coefficients of production, a sequence of $Q_t = [G_t^a, G_t^b, A_t, B_t]$, $P_t = [p_{gt}, p_{at}, p_{bt}, w_t]$, i.e. $\{Q_t; P_t\}_{t \in Z}$, is an equilibrium path for the economy under consideration if and only if it satisfies the following conditions:

$$G_t^a + G_t^b = [g_{1t}(.) + g_{2t}(.)] \cdot L \tag{8}$$

$$L \geq a_{g\ell} G_t^a + a_{a\ell} A_t + b_{g\ell} G_t^b + b_{b\ell} B_t \quad \text{with "=" if } w_t > 0 \tag{9}$$

$$A_{t-1} = a_{ga} G_t^a + a_{aa} A_t \tag{10}$$

$$B_{t-1} = b_{gb} G_t^b + b_{bb} B_t \tag{11}$$

$$p_{gt} \leq a_{ga}\, p_{at-1} + a_{g\ell}\, w_t \quad \text{with "=" if } G_t^a > 0 \tag{12}$$

$$p_{gt} \leq b_{gb}\, p_{bt-1} + b_{g\ell}\, w_t \quad \text{with "=" if } G_t^b > 0 \tag{13}$$

$$p_{at} \leq a_{aa}\, p_{at-1} + a_{a\ell}\, w_t \quad \text{with "=" if } A_t > 0 \tag{14}$$

$$p_{bt} \leq b_{bb}\, p_{bt-1} + b_{b\ell}\, w_t \quad \text{with "=" if } B_t > 0 \tag{15}$$

Equations (8)–(11) do not deserve particular explanation; they are the market-clearing conditions. Equations (12)–(15) come from extra-profits maximization under competitive conditions. First, they rule out price systems which give rise to strictly positive extra-profits, because they are incompatible with competitive equilibrium. Second, they imply that in equilibrium extra-profits must be zero, and then the activity level sequence $Q_t$ is such as to maximize extra-profits at the equilibrium prices (since extra-profits cannot be strictly positive, they are maximized when they are zero).

Noting that, in our economy, wheat is demanded without satiation, its equilibrium prices (one for each date) cannot be zero. Thanks to this fact, we may rewrite conditions (12)–(15) in a more familiar way.[5] In fact, dividing both sides of inequalities (12)–(15) for $p_{gt}$, and remembering that $p_{gt} = p_{gt-1}/(1 + r_t)$, we get:

$$1 \leq a_{ga}\frac{p_{at-1}}{p_{gt-1}}(1 + r_t) + a_{g\ell}\frac{w_t}{p_{gt}} \qquad \text{with ``='' if} \quad G_t^a > 0 \tag{12'}$$

$$1 \leq b_{gb}\frac{p_{bt-1}}{p_{gt-1}}(1 + r_t) + b_{g\ell}\frac{w_t}{p_{gt}} \qquad \text{with ``='' if} \quad G_t^b > 0 \tag{13'}$$

$$\frac{p_{at}}{p_{gt}} \leq a_{aa}\frac{p_{at-1}}{p_{gt-1}}(1 + r_t) + a_{a\ell}\frac{w_t}{p_{gt}} \qquad \text{with ``='' if} \quad A_t > 0 \tag{14'}$$

$$\frac{p_{bt}}{p_{gt}} \leq b_{bb}\frac{p_{bt-1}}{p_{gt-1}}(1 + r_t) + b_{b\ell}\frac{w_t}{p_{gt}} \qquad \text{with ``='' if} \quad B_t > 0 \tag{15'}$$

When we get to this point, as Samuelson observed in his 1958 article, studying the equilibrium path as just defined with the methods used for equilibrium over a finite arc of time (as in the Arrow–Debreu model) is impossible. In fact, however we take an arc of time composed of a finite number $T$ of periods, the system of equilibrium conditions over $T$ will certainly have more unknowns[6] than equations, or using Samuelson's own words, there never seem to be enough equations:

> if we take any finite stretch of time and write out the equilibrium conditions, we always find them containing discount rates from before the finite period and discount rates from afterward. We never seem to get enough equations: lengthening our time period turns out always to add as many new unknowns as it supplies equations ...
> We can try to cut the Gordian knot by our special assumption of stationariness (Samuelson, 1958, p. 470).

So, following Samuelson's advice and a practice by now consolidated in the literature on models with overlapping generations, we too will now focus our attention on stationary equilibrium solutions.

## V. THE CONDITIONS FOR STATIONARY EQUILIBRIUM

A quantity vector: $Q = [G^a, G^b, A, B]$, a vector of prices in terms of wheat: $P = [p_a, p_b, \omega]$ and a rate of interest denominated in wheat $r$, represent a stationary equilibrium[7]

[5] Rewriting conditions (12)–(15) will also make more transparent the linkage with their stationary equilibrium analogous, i.e. conditions (20)–(23) in Section VI.

[6] In our model, the unknowns are prices and produced quantities at each date, while in Samuelson's quoted passage the unknowns are discount rates, one for each date.

[7] It might be useful to recall that in a stationary equilibrium $p_{jt}/p_{gt} = p_j$ and $p_{it}/p_{it+1} = 1 + r$ for every $t$, with $j = a, b$ and $i = g, a, b$.

for the economy under consideration if and only if:

$$G^a + G^b = [g_1(.) + g_2(.)] \cdot L \tag{16}$$

$$L \geq a_{g\ell}\, G^a + a_{a\ell}\, A + b_{g\ell} G^b + b_{b\ell}\, B \quad \text{with “=” if} \quad \omega > 0 \tag{17}$$

$$A = a_{ga}\, G^a + a_{aa}\, A \tag{18}$$

$$B = b_{gb}\, G^b + b_{bb}\, B \tag{19}$$

$$1 \leq a_{ga}\, p_a(1 + r) + a_{g\ell}\, \omega \quad \text{with “=” if} \quad G^a > 0 \tag{20}$$

$$1 \leq b_{gb}\, p_b(1 + r) + b_{g\ell}\, \omega \quad \text{with “=” if} \quad G^b > 0 \tag{21}$$

$$p_a \leq a_{aa}\, p_a(1 + r) + a_{a\ell}\, \omega \quad \text{with “=” if} \quad A > 0 \tag{22}$$

$$p_b \leq b_{bb}\, p_b(1 + r) + b_{b\ell}\, \omega \quad \text{with “=” if} \quad B > 0 \tag{23}$$

As is well known, a characteristic of stationary equilibria is the absence of net savings (or net savings per worker in a steady growth equilibrium). This condition, as we will show, is implicit in our system.

Conditions (18)–(23) imply the following equality:

$$G_a + G_b + Ap_a + Bp_b = L\omega + (Ap_a + Bp_b)(1 + r) \tag{24}$$

which is the equality between the value of gross production and the value of gross income.

From the individual budget constraints, referred to a single period,[8] we have that gross income must be equal to the sum of expenditure for consumption plus gross savings. In other terms:

$$L\omega + (Ap_a + Bp_b)(1 + r) = g_1(.) \cdot L + g_2(.) \cdot L + [\omega - g_1(.)] \cdot L \tag{25}$$

Note that this equation is nothing but Walras's law applied to our case of stationary equilibrium with overlapping generations. Consequently when conditions (18)–(23) are satisfied, Walras's Law implies:

$$G_a + G_b + Ap_a + Bp_b = g_1(.) \cdot L + g_2(.) \cdot L + [\omega - g_1(.)] \cdot L \tag{26}$$

At this point, as can be easily verified, condition (16) implies equality between gross savings and the value of capital employed:

$$L[\omega - g_1(.)] = p_a A + p_b B \tag{27}$$

More precisely, we have that, because of equation (26), condition (16) can be satisfied if and only if condition (27) is satisfied; in other words, these two equations express the same equilibrium condition. For this reason, in our equilibrium system,

[8] It should be noted that the budget constraint in equation (2) is intertemporal, i.e. it refers to both periods of life, and results from the member-to-member summation of the budget constraints of each period expressed in present value.

condition (16) can be replaced by condition (27) without altering the system's significance.

## VI. STUDYING STATIONARY EQUILIBRIUM SOLUTIONS

Study of the solutions to the system (27), (17)–(23) can be effected with a two-stage method (cf. Garegnani 2003). In the first, the interest rate $r$ is considered as an independent variable, and conditions (17)–(23) are seen as a system of equations implicitly defining the price vector $P$ and the quantity vector $Q$ as functions (or correspondences) of the interest rate $r$. Once $P(r)$ and $Q(r)$ have been found, they are used to calculate the behaviour of saving and investment as the interest rate varies. Then, in the second stage, we determine the general equilibrium values of $r$ through equation (27), i.e. the equilibrium condition between saving and investment. Let $r^\star$ be a general equilibrium value of the interest rate, then $P(r^\star)$, $Q(r^\star)$ and $r^\star$ are a solution to system (27), (17)–(23).

### *VI.a. The wage rate as a function of r*

We know that equations (20)–(23) allow us to establish the behaviour of price vector $P$ as the interest rate $r$ varies (cf. Garegnani, 1970).

In particular, from conditions (20) and (22) we get:

$$\omega \geq \omega^a(r) = \frac{1 - a_{aa}(1+r)}{a_{g\ell} + (a_{ga}a_{a\ell} - a_{aa}a_{g\ell})(1+r)} \quad \text{with “=” if} \quad G^a \cdot A > 0 \tag{28}$$

where $\omega^a\,(r)$ is the wage rate as a function of the interest rate if the technique in use is the one employing capital good [A] (technique [A] hereafter).

Similarly, from conditions (21) and (22), we arrive at the following equation:

$$\omega \geq \omega^b(r) = \frac{1 - b_{bb}(1+r)}{b_{g\ell} + (b_{gb}b_{b\ell} - b_{bb}b_{g\ell})(1+r)} \quad \text{with “=” if} \quad G^b \cdot B > 0 \tag{29}$$

Each function $\omega^i\,(r)$, with $i = a,\ b$, is continuous and monotonically decreasing for every economically interesting $r$; i.e. for every $0 \leq r \leq R^i$, where $R^i$ is such that $\omega^i\,(R^i) = 0$.

Given a certain level of the rate of interest , and given the conditions (28) and (29), the technique in use will be the one that permits the highest wage rate[9] (cf. Garegnani, 1970, p. 411, n. 1), therefore we have:

$$\omega = \omega(r) = \max\,\{\omega^a(r),\omega^b(r)\} \tag{30}$$

[9] When the wage rate is set equal to the highest between $\omega^a(r)$ and $\omega^b(r)$, the use of the technique with the highest wage rate will balance the budget, while the use of the other will result in losses. In the opposite case, i.e. when the wage rate is set equal to the lowest between $\omega^a(r)$ and $\omega^b(r)$, the technique with the lowest wage rate balances the budget, while the other will obtain extra-profits. In both cases, all businesses will adopt the technique with the highest wage rate.

As is well known, there will be, in general, at least one interest rate level such that $\omega^a$ and $\omega^b$ are equal. This level of $r$, commonly called "switch point", is a solution of the following equation:

$$\frac{1 - a_{aa}(1+r)}{a_{g\ell} + (a_{ga}a_{a\ell} - a_{aa}a_{g\ell})(1+r)} = \frac{1 - b_{bb}(1+r)}{b_{g\ell} + (b_{gb}b_{b\ell} - b_{bb}b_{g\ell})(1+r)} \tag{31}$$

Since equation (31) is of the second degree, it may have two solutions, and both solutions might be in the range between 0 and $min\{R^a, R^b\}$. In this case—which is what we will assume in the following—there is the phenomenon called "reswitching of techniques".

In particular, assuming that $r'$ and $r''$ are the two strictly positive solutions of equation (31) and $r' < r'' < R^b < R^a$, we may have:

- $\omega^a\ (r) > \omega^b\ (r) \quad \forall 0 \leq r < r'$;
- $\omega^a\ (r) < \omega^b\ (r) \quad \forall r' < r < r''$;
- $\omega^a\ (r) > \omega^b\ (r) \quad \forall r'' < r \leq R^a$ with $R^a$: $\omega^a\ (R^a) = 0$.

In this case, the wage rate as a function of the interest rate is:

$$\omega(r) = \begin{cases} \omega^a(r) & \text{when } r \in [0, r') \\ \omega^{\mathrm{a}}(r) = \omega^b(r) & \text{when } r = r' \\ \omega^b(r) & \text{when } r \in (r', r'') \\ \omega^{\mathrm{a}}(r) = \omega^b(r) & \text{when } r = r'' \\ \omega^a(r) & \text{when } r \in (r'', R^{\mathrm{a}}] \end{cases} \tag{32}$$

Such a function is continuous for every interest rate $r \in [0, R^a]$, and differentiable on its domain with the exception of $r'$ and $r''$.

*VI.b. The quantities as functions of r*

Equilibrium conditions (18) and (19) imply that $G^a = 0$ if and only if $A = 0$ and, similarly, $G^b = 0$ if and only if $B = 0$. This has a very simple economic meaning: when the technique employing capital goods of a certain kind is not in use, there is no need for the production of these capital goods and, on the other side, when the (circulating) capital goods of a certain kind are out of production, the technique employing them cannot be in use in stationary conditions.

On the basis of what we have said above we have that, given a certain interest rate $r$, the technique in use will be [A] or [B] according to $\omega^a(r) > \omega^b(r)$ or $\omega^a(r) < \omega^b(r)$. As a consequence, for every interest rate such that $\omega^a(r) > \omega^b(r)$, we have $G^b(r) = B(r) = 0$, while $G^a(r) = A(r) = 0$ if $\omega^a(r) < \omega^b(r)$.

Taking account of conditions (17)–(19), we can now express the quantities $G^a$ and $G^b$ as correspondences of the interest rate:

$$G^a(r) = \begin{cases} \dfrac{1 - a_{aa}}{a_{g\ell} + (a_{a\ell}a_{ga} - a_{g\ell}a_{aa})} L \ \forall r : \omega^a > \omega^b \\ 0 \ \forall r : \omega^a < \omega^b \\ \left\{ G^a : 0 \leq G^a \leq \dfrac{1 - a_{aa}}{a_{g\ell} + (a_{a\ell}a_{ga} - a_{g\ell}a_{aa})} L \right\} \forall r : \omega^a = \omega^b \end{cases} \tag{33}$$

$$G^b(r) = \begin{cases} \dfrac{1 - b_{bb}}{b_{g\ell} + (b_{b\ell}b_{gb} - b_{g\ell}b_{bb})} L \ \ \forall r : \omega^a < \omega^b \\ 0 \ \forall r : \omega^a > \omega^b \\ \left\{ G^b : 0 \leq G^b \leq \dfrac{1 - b_{bb}}{b_{g\ell} + (b_{b\ell}b_{gb} - b_{g\ell}b_{bb})} L \right\} \forall r : \omega^a = \omega^b \end{cases} \tag{34}$$

*VI.c. The saving function*

With the wage function $\omega(r)$ determined, and given the propensity to save function $s(r)$ defined by equation (5), the saving function is:

$$S(r) = L[\omega(r) - g_1(.)] = L \cdot \omega(r) \cdot s(r) \tag{35}$$

Remembering that, when $r = 0$, technique [A] is the one in use, and denoting with $W^a$ the maximum wage rate,[10] i.e. $W^a = \omega^a(0) = \omega(0)$, we have:

$$S(0) = L \cdot \omega(0) \cdot s(0) = L \cdot W^a \cdot \frac{1}{(1 + \rho)^{1/\theta} + 1} \tag{36}$$

On the other hand, when the interest rate is at its maximum level $R^a$, the wage rate becomes zero and, since workers are the only savers of the economy, savings will be zero too. Thus, $S(R^a) = 0$.

For interest rates between 0 and $R^a$, the amount of saving is strictly positive, and the shape of the saving curve is the result of two conflicting forces. On the one hand, increases in the interest rate will create an increase in young-workers' average propensity to save and, on the other, the increase in the interest rate will be accompanied by a reduction in the wage rate, hence in the income to which the rising propensity to save is applied. Therefore, for a certain interest rate, the saving curve will be rising or decreasing according to whether the elasticity of the wage rate to $r$ is smaller or greater (in absolute value) than the elasticity of the propensity to save.

These two conflicting forces may be seen as a substitution effect and an income effect. In fact, the rising propensity to save, as $r$ increases, is the effect of a substitution between present and future consumption of wheat, while the same change in $r$ also

[10] Comparing equations (28) and (33), we find that $W^a$ equals the net product per worker with technique [A].

brings about an income effect, because of the reduction in the wage rate, and thus in the workers' purchasing power in terms of wheat.

For interest rates close enough to $R^a$, the income effect must prevail, because the saving amount, from strictly positive, must become zero at $R^a$. Therefore, the saving curve must have a final decreasing section. However, we could very well assume that it has also an initial rising section. Setting this assumption seems to us to be a concession to neoclassical theory, because in the rising section of the saving curve the substitution effect, which is the central element of neoclassical theory, prevails over the income effect, which in that theory is seen rather as a collateral, disturbing, effect.

### *VI.d. The investment correspondence*

Given a certain interest rate level, the corresponding demand for capital, in terms of wheat, may be obtained directly from equation (24):

$$K = Ap_a + Bp_b = \frac{1}{r} \cdot (G_a + G_b - L\omega) \tag{24'}$$

In other words, since the difference between net production and the amount of wages is the amount of interest on capital, this difference divided by the interest rate is the quantity of capital employed in terms of wheat, i.e. in value terms.

Denoting with $K^a$ the amount of capital (in terms of wheat) demanded with technique [A], we have that $K^a$ is a function of the interest rate:

$$K^a(r) = L \cdot \frac{W^a - \omega^a(r)}{r} \tag{37}$$

This equation follows directly from (24′), assuming $G_b = B = 0$, and noting that $W^a$ is the net product per worker with technique [A] (see footnote 10), so that $G_a = L \cdot W^a$. The function $K^a(r)$ is continuous and monotonically decreasing[11] for every $r \in [0, R^a]$.

By the same reasoning we get a similar function $K^b(r)$:

$$K^b(r) = L \cdot \frac{W^b - \omega^b(r)}{r} \tag{38}$$

According to our assumption concerning reswitching, we have $K(r) = K^a(r)$ for every $r$ belonging either to $[0, r')$ or to $(r'', R^a]$, while for every $r \in (r', r'')$ we have $K(r) = K^b(r)$. For the levels of interest rate $r'$ and $r''$ $K(r)$ is, instead, set valued, because both the techniques may be in use and the number of workers $L$ may be distributed between them in an infinite set of possible ways. Therefore, we have $K(r') = \{K{:}K = \mu \cdot K^a(r') + (1 - \mu) \cdot K^b(r'),\ \forall 0 \le \mu \le 1\}$ and similarly

[11] The monotonically decreasing shape comes from the assumption: $a_{ga}/a_{g\ell} > a_{aa}/a_{a\ell}$.

$K(r'') = \{K{:}K = \mu \cdot K^a(r'') + (1-\mu) \cdot K^b(r''), \forall 0 \leq \mu \leq 1\}$, where $\mu$ reflects the share of workers employed with the technique using type [A] capital goods.

Summing up, we have the following demand for capital or investment correspondence:

$$K(r) = \begin{cases} K^a(r) & \text{when } r \in [0, r') \\ \{K{:}K = \mu \cdot K^a(r') + (1-\mu) \cdot K^b(r'), \forall 0 \leq \mu \leq 1\} & \text{when } r = r' \\ K^b(r) & \text{when } r \in (r', r'') \\ \{K{:}K = \mu \cdot K^a(r'') + (1-\mu) \cdot K^b(r''), \forall 0 \leq \mu \leq 1\} & \text{when } r = r'' \\ K^a(r) & \text{when } r \in (r'', R^a] \end{cases} \quad (39)$$

We can now easily prove that, when the interest rate rises passing trough $r'$, the demand for capital decreases, while when the interest rate rises passing trough $r''$ the demand for capital increases. This is easily done by observing, first, that because of equations (37) and (38), when $\omega^a = \omega^b$, i.e. when the interest rate is either $r'$ or $r''$, we have $W^a - r \cdot K^a/L = W^b - r \cdot K^b/L$ or, in another form, $W^a - W^b = r \cdot (K^a - K^b)/L$. Therefore, since $W^a > W^b$, we must have $K^a(r') > K^b(r')$ and $K^a(r'') > K^b(r'')$. Finally, the continuity of the functions $K^a(r)$ and $K^b(r)$ implies $K^a(r) > K^b(r)$ for every $r$ in small neighbourhoods of $r'$ and of $r''$. Indeed, we conclude that the first switch of techniques brings about a decrease in the demand for capital, while the second brings about an increase of it.

This conclusion has a corollary which is very important for our aims: even if both $K^a(r)$ and $K^b(r)$ are monotonically decreasing functions, the demand for capital $K(r)$ may increase as the interest rate rises. In particular, this happens when the interest rate rises passing through $r''$ (cf. the numerical example in Section VII). In this case, the increase in the demand for capital is indubitably associated with the phenomenon of the reswitching of techniques.

### *VI.e. The equilibrium levels of r*

Given the saving function (35) and the investment correspondence (39), we can determine the equilibrium levels of $r$. In particular, $r^\star$ is an equilibrium for the model we are considering if and only if:

$$S(r^*) \in K(r^*) \quad (40)$$

One can conceive some set of parameters, i.e. technical coefficients and preference parameters, for which no equilibrium exists. In particular, it may happen that $S(r) < \inf K(r)$ for every $r \in [0, R^a]$. Therefore, in order to avoid this unwelcome result, we must impose some restrictions on admissible parameters.

A sufficient restriction granting the existence of at least one equilibrium is the following. Let us start by considering one technique only, say technique [A]. Given some parameters $\rho$ and $\theta$, we find technological coefficients such that there are some interest rate levels for which $S(r) > K^a(r)$. Since $K^a(r)$ and $S(r)$ are continuous functions and $K^a(R^a) > S(R^a) = 0$, there must exist an $\bar{r}$ such that $S(\bar{r}) = K^a(\bar{r})$.

We now add the second technique, whose technical coefficients must be such that $\omega^a(\bar{r}) > \omega^b(\bar{r})$. In this case, we are guaranteed that at least an interest rate level $\bar{r}$ satisfying condition (40) exists.

The complete set of equilibrium interest rate levels may be identified graphically by super-imposing the savings curve on the investment curve, i.e. respectively, the graph of the function $S(r)$ and that of the correspondence $K(r)$ (cf. Figures 1 and 2 in Section VII). In particular, because of the reverse U shape of the saving curve there are, generically, at least two equilibria; and because of the upward jump of the investment schedule, due to the reswitching, there may be multiple equilibria on the rising stretch of the saving curve. This is proved by the numerical example in the following paragraph.

## VII. A NUMERICAL EXAMPLE

Let us consider a model of the same type as the one described previously. Set the amount of labour available $L$ as equal to 1, and assume the following numerical values for the parameters:

*Preference parameters*

$\theta = 0.08 \quad \rho = 0.8$

*Technical coefficients*

$a_{aa} = 0.15 \quad a_{a\ell} = 0.5 \quad a_{ga} = 0.7 \quad a_{g\ell} = 0.6$

$b_{bb} = 0.254 \quad b_{b\ell} = 0.403 \quad b_{gb} = 0.5 \quad b_{g\ell} = 0.788$

With the above technical coefficients, equation (31) has two solutions: $r' = 0.5$ and $r'' = 0.925$. In this case therefore reswitching occurs, so that the demand for capital schedule, defined by equation (39), makes an upward jump. More precisely, the

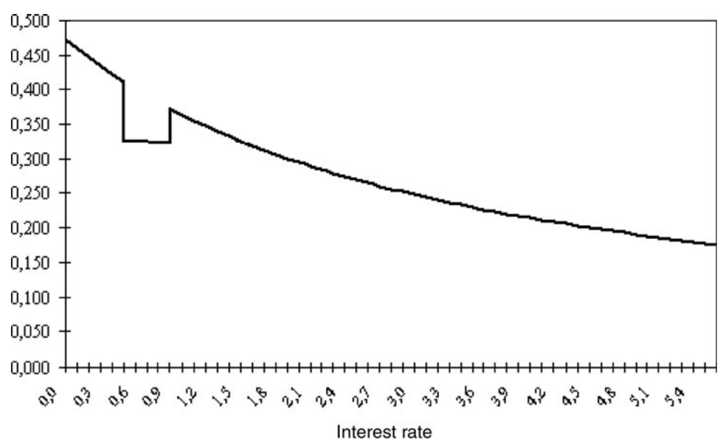


FIGURE 1. Investment schedule.

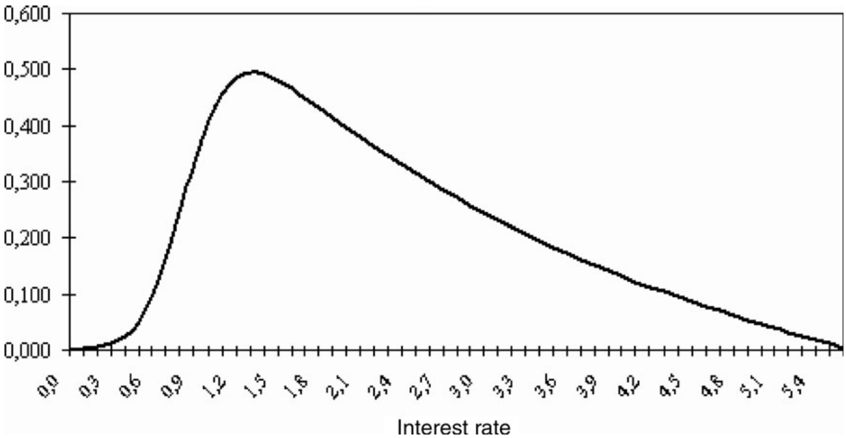


FIGURE 2. Saving schedule.

schedule for the demand for capital (or investment) has the shape shown in Figure 1. We can also plot the saving schedule, which, in the case we are considering, has a reverse U shape, shown in Figure 2.

By superimposing the equilibrium savings curve (Fig. 2) on the equilibrium investment curve (Fig. 1), the levels of the interest rate $r$ at which the economy is in equilibrium could be identified graphically. We prefer to identify them numerically, as in Table 1. We have therefore four equilibria—the corresponding interest rate levels are indicated by bold characters in Table 1—and three of them are in the rising stretch of the saving schedule.

A further noteworthy feature concerns local stability of stationary equilibria. In particular, using standard tools, both the equilibria for $r = 0.89$ and $r = 0.94$ can be proved asymptotically locally stable under at least two dynamic processes.

This can be done by starting from a simple assumption: if at a stationary equilibrium interest rate $r^\star$ only a certain technique is in use, then we assume that for interest rates which are close enough to $r^\star$ the same technique will be in use. This means that for interest rates in a neighbourhood of 0.89 only technique [B] is in use, while for interest rates in a neighbourhood of 0.94 technique [A] is the one in use.

The first dynamic process we consider is the Walrasian tâtonnement, which is defined, in our case with saving and investment, by the following differential equation:

$$dr_\tau/d\tau = h[K^i(r_\tau) - S(r_\tau)] \quad \text{with} \quad i = a, b \tag{41}$$

where $h(\cdot)$ is a sign-preserving differentiable function.

Recalling that $K^i(r)$, with $i = a, b$, is a monotonically decreasing function, and since $S(r)$ is rising for both $r = 0.89$ and $r = 0.94$, we see that the tatônnement is

TABLE 1. Numerical determination of investments and savings.

| Interest rate | Technique in use | Demand for capital (investment) | Savings | $w^a$ | $w^b$ | $K^a$ | $K^b$ | Average propensity to save |
|---|---|---|---|---|---|---|---|---|
| 0.00000 | A | 0.47323 | 0.00064 | 0.98837 | 0.94508 | 0.47323 | 0.32340 | 0.00064 |
| 0.20000 | A | 0.44625 | 0.00469 | 0.89912 | 0.88043 | 0.44625 | 0.32329 | 0.00522 |
| 0.40000 | A | 0.42218 | 0.02454 | 0.81950 | 0.81581 | 0.42218 | 0.32318 | 0.02995 |
| 0.48075 | A and B | [0.32313; 0.41318] | 0.04388 | 0.78974 | 0.78974 | 0.41318 | 0.32313 | 0.05557 |
| 0.60000 | B | 0.32307 | 0.09420 | 0.74803 | 0.75124 | 0.40057 | 0.32307 | 0.12540 |
| 0.80000 | B | 0.32296 | 0.24526 | 0.68352 | 0.68672 | 0.38106 | 0.32296 | 0.35714 |
| **0.88816** | B | 0.32291 | 0.32291 | 0.65704 | 0.65829 | 0.37306 | 0.32291 | 0.49053 |
| **0.92842** | A and B | [0.32289; 0.36951] | 0.35557 | 0.64531 | 0.64531 | 0.36951 | 0.32289 | 0.55100 |
| **0.94455** | A | 0.36811 | 0.36811 | 0.64067 | 0.64011 | 0.36811 | 0.32288 | 0.57457 |
| 1.00000 | A | 0.36337 | 0.40693 | 0.62500 | 0.62224 | 0.36337 | 0.32285 | 0.65110 |
| 1.20000 | A | 0.34725 | 0.48485 | 0.57167 | 0.55780 | 0.34725 | 0.32274 | 0.84812 |
| 1.50000 | A | 0.32558 | 0.48023 | 0.50000 | 0.46123 | 0.32558 | 0.32257 | 0.96046 |
| 2.00000 | A | 0.29491 | 0.39654 | 0.39855 | 0.30049 | 0.29491 | 0.32230 | 0.99497 |
| **2.95290** | A | 0.25002 | 0.25002 | 0.25008 | — | 0.25002 | — | 0.99979 |
| 3.20000 | A | 0.24053 | 0.21865 | 0.21868 | — | 0.24053 | — | 0.99989 |

asymptotically stable[12] for every initial interest rate level $r_0$ in a neighbourhood of 0.89 or of 0.94.

The second dynamic process is that used by Diamond (1965) in order to build a non-stationary path (cf. also Geanakoplos 1980). Diamond's idea is the following: given the rate of interest $r_t$, and the technique in use in period $t$, we may determine the quantity of capital goods produced in that period and their price; we can consequently determine the rate of interest $r_{t+1}$ for which the savings, in period $t$, are equal to the value of capital goods produced. We have, therefore, the difference equation:

$$K^i(r_t) = \omega^i(r_t) \cdot s(r_{t+1}) \cdot L \quad \text{with} \quad i = a, b \tag{42}$$

With the data of our numerical example, the traditional "cobweb theorem" implies[13] that, for any initial interest rate level $r_0$ in a neighbourhood of 0.89 or of 0.94, the dynamic process described by equation (42) is asymptotically stable.

## VIII. THE RESWITCHING OF TECHNIQUES AND THE MULTIPLICITY OF EQUILIBRIA

The multiplicity of solutions is certainly not surprising for a general equilibrium model, so the contribution offered by this paper does not concern multiplicity as such, but rather what causes it.

For the model we are dealing with, if it admits stationary equilibria, there are, generically, at least two equilibria. This can be proved easily. Let us imagine there is only one technique and that it employs wheat, i.e. the consumption good, as capital. In this case, the investment curve is a horizontal line, while, from the argument in Section VI.c, the saving curve has a reversed U shape. If a horizontal line and a bell-shaped curve intersect, there are generically two intersections: one on the rising stretch of the savings curve and one on its decreasing stretch.

Therefore, reswitching of techniques is not necessary for multiplicity. However, in the case considered here, reswitching is necessary for a multiplicity of equilibria in the rising stretch of the savings curve. In fact, if the investment curve had a non-increasing shape, then only one equilibrium on the rising stretch of the saving curve would be possible. Moreover, since we are assuming that the production of both types of capital goods is less capital intensive than the production of wheat, the non-monotonic shape of the investment curve is possible if and only if reswitching occurs.

12 As usual, a dynamic process is asymptotically locally stable if for any $x_0$ in a neighbourhood of $x^\star$ we have $x(\tau,x_0) \to x^\star$ as $\tau \to \infty$ (we use $\tau$ for continuous time, and $t$ for discrete time periods).

13 Let us denote by $\kappa(r_t)$ the value of capital produced in $t$ expressed in "labour commanded", i.e. $\kappa(r_t) = K^i(r_t)/\omega^i(r_t)$. Recalling that $L = 1$, equation (42) can then be written as $\kappa(r_t) = s(r_{t+1})$. Now, for the "cobweb theorem", the dynamic process is (asymptotically) stable if $|\mathrm{d}\kappa/\mathrm{d}r| < |\mathrm{d}s/\mathrm{d}r|$, which is exactly what we have, with the data of our numerical example, for interest rate levels in a neighbourhood of 0.89 or of 0.94. In particular, for $r = 0.89$ we have $\mathrm{d}s/\mathrm{d}r = 1.52$ and $\mathrm{d}\kappa/\mathrm{d}r = 0.24$, while for $r = 0.94$ we have $\mathrm{d}s/\mathrm{d}r = 1.45$ and $\mathrm{d}\kappa/\mathrm{d}r = 0.12$.

When multiplicity of equilibria is caused by an inversion of capital deepening due to reswitching, its characteristics are different from the cases discussed by Samuelson (1958) and in subsequent literature. A couple of remarks will make this point clearer.

First, in our example the stationary equilibrium at the lowest interest rate is the one with the lowest net product per worker, and then equilibria with higher interest rates entail a higher net product per worker.[14] A similar result is certainly impossible in cases where the inversion of capital deepening has no influence on multiplicity. In fact, if no problems of reswitching and reverse capital deepening were to arise, equilibria with higher interest rates would entail either the same or a lower net product per worker.

Second, the multiple equilibria on the rising stretch of the saving curve cannot be dismissed by a stability argument. As we have proved above (Section VII), in our example there are two equilibria on the rising stretch of the saving curve which are locally stable with respect both to the Walrasian tâtonnement and to Diamond's dynamic process. The model is therefore unable to foresee which one of those two equilibrium configurations will be reached,[15] because they seem to be both equally theoretically possible.

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[14] It is worth recalling that, with two techniques, the technique with the highest net product per worker entails also the highest value of capital per worker at a switch point (cf. Section VI.d).

[15] For this observation the author is indebted to an anonymous referee.